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A stable representation of the transmittance between identical circular apertures

Q. Gueuning, C. Craeye and C. Oestges
Institute of Information and Communication Technologies, Electronics and Applied Mathematics (ICTEAM), Universite Catholique de Louvain, Belgium

Abstract

In recent work, the transmittance between circular apertures was formulated, thanks to the projection of equivalent currents evaluated on the apertures over Zernike polynomials, through a polynomial series of a near-field coefficient which is accurate for large apertures and low-order modes. It was observed that this expansion becomes numerically instable for low relative distances. In this paper, we derive a stable representation which can be calculated at arbitrary distance. In case of identical modes, it reduces to a closed-form expression for which an asymptotic near-field formula is given.

1 Introduction

Orbital Angular Modulation (OAM) has given a new momentum to the study of coupling between large aperture antennas. Indeed, in [1], based on experimental observations, it was claimed that the exploitation of OAM, i.e. aperture modes with $e^{jn\phi}$ dependence (ϕ is azimuthal angle, l is an integer), could lead to communication through an arbitrary number of modes within a given frequency band. The limitations of this process could be rapidly guessed from the conical pattern obtained for larger orders n. From there, it was soon observed that the transmittance between OAM antennas undergo a faster decay versus distance for high orders, of type $1/r^{n+1}$, which essentially limits the exploitation of OAM to the near (and maybe intermediate) field region. This has been made quantitatively clearer in [3], where aperture fields are decomposed into Zernike functions to provide a simple spectral expression for transmittance, as well as an intermediate-to-far field asymptotics for transmittance, while also including the radial variation of the field distribution. A higher-order development was presented in [4], to allow a fast estimation of the transmittance between any pair of apertures, down into the intermediate field. When identical modes are considered for the transmitting and receiving antennas, a highly characteristic feature of the transmittance corresponds to the nearly constant - gently undulating - transmittance versus distance, corresponding to the tubular radiation pattern of aperture antennas in the near field. This allows the definition of two clearly differerent regimes of operation, for far and near fields. Asymptotic expressions of the transmittance have already been developed in [3] for the far-field case.

In this paper, the transmittance for apertures of same size is represented through a hypergeometric function of a near-field parameter, thereby extending the approach in [4]. For identical modes, the expression reduces to a sum of modulated Bessel functions. At low distances, it is proven that the transmittance converges to a constant value. The paper is organized as follows. The mathematical expressions of the transmittance are presented in Section 1. Then, we will describe the numerical results in Section 2. Finally, a conclusion is drawn in Section 3.

2 Mathematical formulation

In [3], the transmittance H between two aperture antennas is related to a reaction integral between equivalent current distributions evaluated on the transmitting and receiving apertures. Those currents are then projected onto separable functions which are orthogonal over the unit disk and correspond to the products between a modified circle polynomials of order m of the radial coordinate and a harmonic function $e^{jn\alpha}$ of the azimuthal coordinate. Subsequently, the transmittance is decomposed as [3]

$$H = \sum_n \sum_m \sum_{m'} \gamma_{nm}\, \gamma_{nm'}\, H_{nm'm'} \tag{1}$$

where n is an azimuthal index, m and m' are, respectively, radial indices of the transmitting and receiving current distributions and γ_{nm}, $\gamma_{nm'}$ are the projection coefficients of those currents. Considering apertures of radius a separated by a relative distance z, the quantity $H_{nmm'}$ can be renamed $\mathscr{H}$ and expressed in the spectral-domain as [3]

$$\mathscr{H} = \frac{-4k\eta\pi a^2}{I_r I_t} \int_0^\infty J_s(\beta a)\, J_{s'}(\beta a)\, \frac{e^{-jk_z z}}{\beta\, k_z}\, \mathrm{d}\beta \tag{2}$$

where k and η are the free-space wavenumber and impedance, I_t is the port current of the transmitting antenna, J_s the sth order Bessel function of the first kind with $s = |n| + 2m + 1$ and $k_z = (k^2 - \beta^2)^{1/2}$ with radial wavenumber β.

For intermediate to far distances z and moderate indices n and m, the spectrum of the current distributions decreases faster than that of the Green's function. Hence, the integrand of (2) can be truncated at $|k_z| \ll k$ and we can use a quadratic approximation $k_z \approx k - j\beta^2/(2k)$ in the phase exponent and $k_z \approx k$ in the denominator. After a change of variables $\beta' = \beta a$, we have

$$\mathscr{H} \approx L\, e^{-jkz} \int_0^\infty J_s(\beta a)\, J_{s'}(\beta a)\, \frac{e^{j\beta^2/(2\varepsilon)}}{\beta}\, \mathrm{d}\beta \qquad (3)$$

where the near-field coefficient is $\varepsilon = ka^2/z = \pi z/(4z_f)$, the far-field distance is $z_f = 8a^2/\lambda$ and we substitute $L = -4\eta\pi a^2/(I_r I_t)$. In [4], this integral was first decomposed into a polynomial series and then integrated analytically term by term. Equivalently, it can be rewritten in terms of a generalized hypergeometric function with the help of [5],

$$\mathscr{H} = L\, C_{s,s'}\, e^{-jkz}\, \varepsilon^q\, {}_3F_3\left[\begin{matrix} q+1/2, q+1, q \\ s+1, s'+1, 2q+1 \end{matrix}; -2j\varepsilon\right] \quad (4)$$

where the factors are defined as $q = (s+s')/2$, $C_{s,s'} = q!(j/2)^q/((s+1)!(s'+1)!)$. The hypergeometric function can be found in most mathematical libraries, such as the Matlab software [6].

In case of identical modes $s = s'$, which are dominant in the near-field, the hypergeometric function can be simplified to a closed-form expression [7],

$$\mathscr{H}_{s=s'} = -\frac{L\, e^{-jkz}}{2s}\left(e^{-j\varepsilon}\sum_{i=0}^{s} b_i\, j^i\, J_i(\varepsilon) - 1\right) \qquad (5)$$

where $b_0 = 1, b_1 = 2, ..., b_{s-1} = 2, b_s = 1$. In the near-field region, i.e. for $\varepsilon \gg 1$, we can use the large argument approximation of the Bessel functions to obtain the formula,

$$\mathscr{H}_{s=s'} \sim -\frac{L\, e^{-jkz}}{2}\; \left((1-j)/\sqrt{\pi\varepsilon} - 1/s\right) \qquad (6)$$

This asymptotic expression can be used for $\varepsilon \ll s^2/\pi$ and indicates that the amplitude of the transmittance converges to $L/(2s)$ for null distances z and then slightly decreases for larger distances z.

3 Numerical results

For the following numerical experiments, we will consider a wavelength of 1 cm and an aperture diameter of 40λ . Firstly, the transmittance is illustrated in Fig.1 for $n = 0, ..., 4$, $m = 2$ and $m' = 1$. One can observe that the model (4) is close to the exact numerical integration of (2) down to approximatively 2 digits of accuracy over the whole range of relative distances z. Expression (4) is thus numerically stable, in contrast with the direct evaluation of the series representation proposed in [4] which appears to diverge around $z/z_f \approx 10^{-2}$. One must note that the hypergeometric function is mathematically defined by an infinite polynomial series in ε, similarly to [4]. However, it has been observed that the evaluation of the hypergeometric function with a built-in Matlab routine is stable and that the resulting computation time does not depend on its argument.

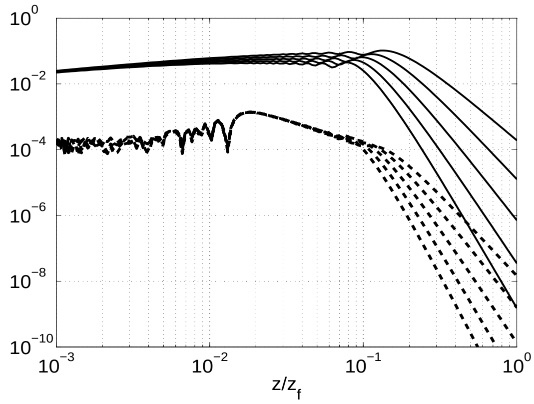

Figure 1. Transmittance and absolute error curves for $m = 2$ $m' = 1$ and n from 0 to 4 (top to bottom). Numerical integration of (2) (solid line) and absolute error of the expression (4) (dashed line).

Secondly, the near-field approximation (6) is validated in Fig.2 for $n = 0, 4$ and $m = m' = 2$. Indeed, the transmittance converges to a constant amplitude $L/(2s)$ for low distances z. One can also infer that the combination of the near-field asymptotic and the far-field expression (17) in [3] could provide a useful tool in the form of a low-pass filter for the design of aperture antennas.

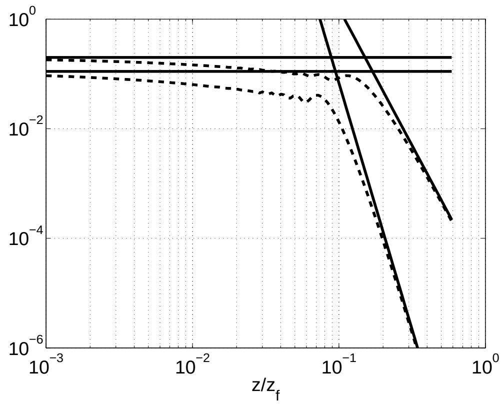

Figure 2. Transmittance and asymptotics for $m = m' = 2$ and $n = 0, 4$ (top to bottom).

4 Conclusion

We have presented mathematical expressions for the transmittance between apertures of same size that can be evaluated at arbitrary relative distances. For the given example, it has achieved 2 digits of accuracy compared to an exact numerical integration. We have also derived a near-field asymptotic formula for identical modes. Future work will focus on the quantification of the resulting error.

References

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