Innovating SI Units in Maxwell's Equations. Evolutionary Approach to Electrodynamics as an Alternative to the Time-Harmonic Field Concept

(Invited Paper)

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***Abstract*—A novel format of Maxwell's equations in SI units is proposed. To this aim, the original field vectors, $\mathcal{E}$ and $\mathcal{H}$, are scaled by the specially introduced factors, carrying dimension of volt and ampere, respectively. Maxwell's equations in novel format involve only one dimensional physical constant, namely: the speed of light in vacuum. Maxwell's equations in that format are applied for studying propagation in waveguides of the time-varying waveforms (signals). To this aim, a simplified waveguide version of the evolutionary approach to electrodynamics is presented. Ultimately, modal amplitudes of the signals are expressed via solutions to Klein-Gordon equation, the arguments of which involve axial coordinate z and time t as variables.**

I. Introduction

Ever since the SI metric system has been introduced in general usage, the electric and magnetic field vectors, $\mathcal{E}$ and $\mathcal{H}$, carry units of *volt per meter*, $\lfloor \mathrm{V\,m^{-1}} \rfloor$ and *ampere per meter*, $\lfloor \mathrm{A\,m^{-1}} \rfloor$, respectively. That is why the free space constants, ϵ_0 and μ_0, with their dimensions of *farad per meter*, $\lfloor \mathrm{F\,m^{-1}} \rfloor$, and *henry per meter*, $\lfloor \mathrm{H\,m^{-1}} \rfloor$, have been *empirically* installed in Maxwell's equations [1]. Eventually, these equations appear in their present form as

$$\nabla\times \underbrace{\mathcal{H}(\mathbf{r},t)}_{\lfloor \mathrm{A\,m^{-1}} \rfloor} = \underbrace{\epsilon_0}_{\lfloor \mathrm{F\,m^{-1}} \rfloor} \frac{\partial}{\partial t} \underbrace{\mathcal{E}(\mathbf{r},t)}_{\lfloor \mathrm{V\,m^{-1}} \rfloor} + \underbrace{\mathcal{J}(\mathbf{r},t)}_{\lfloor \mathrm{A\,m^{-2}} \rfloor} \tag{1a}$$

$$\nabla\times \underbrace{\mathcal{E}(\mathbf{r},t)}_{\lfloor \mathrm{V\,m^{-1}} \rfloor} = - \underbrace{\mu_0}_{\lfloor \mathrm{H\,m^{-1}} \rfloor} \frac{\partial}{\partial t} \underbrace{\mathcal{H}(\mathbf{r},t)}_{\lfloor \mathrm{A\,m^{-1}} \rfloor} \tag{1b}$$

$$\epsilon_0 \nabla\cdot\mathcal{E}(\mathbf{r},t) = \rho(\mathbf{r},t), \quad \mu_0 \nabla\cdot\mathcal{H}(\mathbf{r},t) = 0; \tag{1c}$$

the free-space constants have been specified in [1] as

$$\epsilon_0 = \frac{1}{\mathrm{c}_0^2\, 4\pi\times 10^{-7}} \lfloor \mathrm{F\,m^{-1}} \equiv \mathrm{A^2\,s^2\,N^{-1}\,m^{-2}} \rfloor \tag{2a}$$

$$\mu_0 = 4\pi\times 10^{-7} \lfloor \mathrm{H\,m^{-1}} \equiv \mathrm{N\,A^{-2}} \rfloor. \tag{2b}$$

Herein, $\mathrm{N} \equiv \mathrm{kg\,m\,s^{-2}}$ is the force unit, *newton*, and *number* $\mathrm{c}_0 = 2.99792458\times 10^8$ is the *quantity symbol* for the speed of light in vacuum. That is, $c = 1/\sqrt{\epsilon_0\mu_0} = \mathrm{c}_0 \lfloor \mathrm{m\,s^{-1}} \rfloor$.

Habitual equations $(1a) - (1c)$ are overloaded by various physical dimensions of the electromagnetic field quantities and dimensional constants, which are present therein. Visually, they look as a zoo with the physical dimensions. In the next section, I propose a simpler format of Maxwell's equations, but staying within the framework of the SI metric system.

II. New Format of Maxwell's Equations in SI Units

Analysis what follows is an extension of my presentation given recently in [2]. Combination $\sqrt{\mathrm{N}/(\mathrm{H\,m^{-1}})}$ of the force unit, N, and the dimension of μ_0, i.e., $\lfloor \mathrm{H\,m^{-1}} \rfloor$, yields *ampere*, $\lfloor \mathrm{A} \rfloor$. Combination of N and the dimension of ϵ_0, i.e., $\lfloor \mathrm{F\,m^{-1}} \rfloor$, yields $\sqrt{\mathrm{N}/(\mathrm{F\,m^{-1}})} = \lfloor \mathrm{N\,m\,A^{-1}\,s^{-1}} \rfloor$ what is the dimension of *volt*, $\lfloor \mathrm{V} \rfloor$. This observation suggests to *define* a pair of new dimensional constants, which we denote as ϵ_0^{V} and μ_0^{A} and present them as follows:

$$\epsilon_0^{\mathrm{V}} \stackrel{def.}{=} \sqrt{\mathrm{N}/\epsilon_0} \cong 3.3607\times 10^5 \lfloor \mathrm{V} \rfloor \tag{3a}$$

$$\mu_0^{\mathrm{A}} \stackrel{def.}{=} \sqrt{\mathrm{N}/\mu_0} \cong 8.9206\times 10^2 \lfloor \mathrm{A} \rfloor. \tag{3b}$$

I propose to *scale* the original field vectors, $\mathcal{E}$ and $\mathcal{H}$, with using these new constants as the scaling factors, that is,

$$\boxed{\underbrace{\mathcal{E}(\mathbf{r},t)}_{\lfloor \mathrm{V\,m^{-1}} \rfloor} = \underbrace{\epsilon_0^{\mathrm{V}}}_{\lfloor \mathrm{V} \rfloor} \underbrace{\mathbb{E}(\mathbf{r},t)}_{\lfloor \mathrm{m^{-1}} \rfloor}, \quad \underbrace{\mathcal{H}(\mathbf{r},t)}_{\lfloor \mathrm{A\,m^{-1}} \rfloor} = \underbrace{\mu_0^{\mathrm{A}}}_{\lfloor \mathrm{A} \rfloor} \underbrace{\mathbb{H}(\mathbf{r},t)}_{\lfloor \mathrm{m^{-1}} \rfloor}} \tag{4}$$

where the new field vectors, $\mathbb{E}$ and $\mathbb{H}$ both, carry their simple common physical dimension of *inverse meter*, $\lfloor \mathrm{m^{-1}} \rfloor$.

The electric current density, $\mathcal{J}$, which is present in Eq. $(1a)$, should be scaled as $\mathcal{J} = \mu_0^{\mathrm{A}}\,\mathbb{J}$ where vector $\mathbb{J}$ carries dimension of $\lfloor \mathrm{m^{-2}} \rfloor$. Sometimes, it is proper to present that vector, $\mathcal{J}$, via Ohm's law as $\mathcal{J} = \sigma\,\mathcal{E}$ where a given constant conductivity σ carries dimension of *siemens per meter*, $\lfloor \mathrm{S\,m^{-1}} \rfloor$. In this case, $\mathcal{J} = \mu_0^{\mathrm{A}}\,\mathbb{J}$ where $\mathbb{J} = \gamma_0\,\mathbb{E}$ and $\gamma_0 = 376.73\,\sigma_0$. Factor γ_0 carries dimension of *inverse meter*, $\lfloor \mathrm{m^{-1}} \rfloor$, provided that *number* σ_0 is *quantity symbol* of σ (i.e., σ_0 is a *number* of $\lfloor \mathrm{S\,m^{-1}} \rfloor$ in given σ). Substitutions of the formulas for field scaling (4) and $\mathcal{J} = \mu_0^{\mathrm{A}}\,\mathbb{J}$ to Maxwell's equations $(1a) - (1c)$ result in

$$\nabla\times\mathbb{H}(\mathbf{r},t) = \frac{1}{c}\frac{\partial}{\partial t}\mathbb{E}(\mathbf{r},t) + \mathbb{J}(\mathbf{r},t) \tag{5a}$$

$$\nabla\times\mathbb{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}\mathbb{H}(\mathbf{r},t) \tag{5b}$$

$$\nabla\cdot\mathbb{E}(\mathbf{r},t) = \varrho(\mathbf{r},t), \quad \nabla\cdot\mathbb{H}(\mathbf{r},t) = 0 \tag{5c}$$

where c is speed of light in vacuum. Charge density ρ in $(1c)$ carries dimension of *coulomb per meter*^{3}, $\lfloor \mathrm{C\,m^{-3}} \rfloor$. We apply scaling $\rho = \sqrt{\mathrm{N}\epsilon_0}\,\varrho$, where ϱ is a new quantity aimed for $(5c)$. The scaling factor, $\sqrt{\mathrm{N}\epsilon_0}$, has numerical value 2.9756×10^{-6} and carries dimension of *coulomb per meter*, $\lfloor \mathrm{C\,m^{-1}} \rfloor$. So, ϱ carries dimension of $\lfloor \mathrm{m^{-2}} \rfloor$ in $(5c)$.

In the proposed new format of Maxwell's equations in SI units, $(5a)-(5c)$, just one, but *fundamental* physical constant participates, namely: speed of light in vacuum, c. Besides, the new field vectors, $\mathbb{E}$ and $\mathbb{H}$, and that constant γ_0 carry their *common* physical dimension of *inverse meter*, $\lfloor \mathrm{m}^{-1} \rfloor$.

Thus, one can operate with our simple format of Maxwell's equations $(5a)-(5c)$. Then original field quantities can be recovered (when needed) by the scaling formulas (4), $\mathcal{J} = \mu_0^{\mathrm{A}}\, \mathbb{J}$, and $\rho = \sqrt{\mathrm{N}\epsilon_0}\, \varrho$. In closing, we present Lorentz force law as being written in terms of the field vectors $\mathbb{E}$ and $\mathbb{H}$. To this aim, we start with its classical definition in SI units as

$$\mathcal{F}(\mathbf{r},t) = q\, \left[\mathcal{E}(\mathbf{r},t) + \mathbf{v} \times \mu_0 \mathcal{H}(\mathbf{r},t)\right] \tag{6}$$

where q is a charge measurable in *coulomb*, $\lfloor \mathrm{C} \equiv \mathrm{A\,s} \rfloor$, which a pointwise charged particle, moving with velocity $\mathbf{v}$, carries. Substitution of (4) to (6) results in

$$\mathbb{F}(\mathbf{r},t) = q\, \epsilon_0^{\mathrm{V}} \left[\mathbb{E}(\mathbf{r},t) + \frac{\mathbf{v}}{c} \times \mathbb{H}(\mathbf{r},t)\right] \tag{7}$$

where factor $q\, \epsilon_0^{\mathrm{V}}$ can be rearranged to $q_0\, 3.3607\times 10^5\, \lfloor \mathrm{C\,V} \rfloor$ with using $(3a)$. Herein, q_0 is *quantity symbol* of charge (i.e., a *number* of *coulomb*s, $\lfloor \mathrm{C} \rfloor$, in given q) and V is dimension of *volt*. Dimension $\lfloor \mathrm{C\,V} \rfloor$ is equivalent to $\lfloor \mathrm{N\,m} \rfloor$. Taking into account that $\mathbb{E}$ and $\mathbb{H}$ carry dimension of $\lfloor \mathrm{m}^{-1} \rfloor$, one can be seen that $\mathbb{F}$ carries SI dimension of force, i.e., *newton* $\lfloor \mathrm{N} \rfloor$.

In the next section, we apply equations $(5a)-(5c)$ to study the propagation in waveguides for the time-varying waveforms (signals). To this aim, Evolutionary Approach to Electrodynamics (EAE) will be used [3]. Scheme of a cavity version of the EAE was presented in [4], [5]. In the next Section, a simplified waveguide version of the EAE will be presented.

III. Outline of a Waveguide Version of the EAE

A waveguide is hollow and geometrically homogenous along its axis Oz. Waveguide cross section, S, is bounded by a closed singly-connected contour, L. The contour is smooth enough in the sense that none of its possible *inner* angles (i.e., measured *within* S) exceeds π. In analysis what follows, a triplet of the mutually orthogonal unit vectors, $\mathbf{z} \times \mathbf{l} = \mathbf{n}$, will be used where unit vector $\mathbf{z}$ is axial, $\mathbf{l}$ and $\mathbf{n}$ are the tangential and normal to L unit vectors, respectively. The waveguide surface is perfectly conducting. Hence, equations $(5a)-(5c)$ should be supplemented with the boundary conditions as

$$\mathbf{l}\cdot\mathbb{E}(\mathbf{r},t)\,|_L{=}0, \quad \mathbf{z}\cdot\mathbb{E}(\mathbf{r},t)\,|_L{=}0, \quad \mathbf{n}\cdot\mathbb{H}(\mathbf{r},t)\,|_L{=}0. \tag{8}$$

Such waveguides can support propagation of the transverse-electric (TE) and transverse-magnetic (TM) fields.

A. Incomplete Separation of Variables in Waveguide Modes

Derivation of the (TE) modal fields starts from definition of a complete set of solutions to the Neumann boundary-eigenvalue problem for the transverse part of Laplacian, that is, $\nabla_\perp^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$, formulated as follows:

$$\nabla_\perp^2 \psi_n + \nu_n^2\, \psi_n{=}0, \quad \mathbf{n}\cdot\nabla_\perp \psi_n|_L{=}0, \quad \frac{\nu_n^2}{S}\int_S \psi_n^2\, ds{=}1 \tag{9}$$

where $\psi_n \equiv \psi_n(\mathbf{r}_\perp)$, $\mathbf{r}_\perp$ is projection of the position vector, $\mathbf{r} = \mathbf{r}_\perp + \mathbf{z}\, z$, onto the cross-section domain, S, and $\nu_n^2 > 0$ are the eigenvalues parameterized by the subscript $({}_n)$ in their increasing numerical values order, $n = 1, 2, \ldots$. Having the solutions $\psi_n(\mathbf{r}_\perp)$ at hand as the potentials, we can introduce the vectorial elements of the modal basis, i.e., $[\nabla_\perp \psi_n \times \mathbf{z}]$, $\nabla_\perp \psi_n$, $[\mathbf{z}\, \nu_n \psi_n]$, and present a complete set of the TE- modal fields in waveguides as

$$\mathbb{E}_n^{\mathrm{TE}} = \mathcal{A}_n(z,t)\, [\nabla_\perp \psi_n \times \mathbf{z}], \quad \mathbb{E}_{z\,n}^{\mathrm{TE}} \equiv 0 \tag{10a}$$

$$\mathbb{H}_n^{\mathrm{TE}} = \mathcal{B}_n(z,t)\, \nabla_\perp \psi_n + h_n(z,t)\, [\mathbf{z}\ \nu_n \psi_n] \tag{10b}$$

where $\mathcal{A}_n$, $\mathcal{B}_n$ and h_n are the modal amplitudes of the transverse and longitudinal field components, which are unknown as yet. Nevertheless, one can verify that the field $(10a)-(10b)$ already satisfies the boundary conditions (8). In order to obtain a problem for the modal amplitudes, $\mathcal{A}_n$, $\mathcal{B}_n$, and h_n, one should first put $\mathbb{J} = 0$ and $\varrho = 0$ in $(5a)-(5b)$, as long as the waveguide is hollow, then rearrange Maxwell's equations to their transverse-longitudinal format and substitute therein the fields $(10a)-(10b)$. Eventually, these manipulations result in Klein-Gordon equation (KGE) for h_n as

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} h_n(z,t) - \frac{\partial^2}{\partial z^2} h_n(z,t) + \nu_n^2\, h_n(z,t) = 0 \tag{11}$$

and direct formulas for the amplitudes $\mathcal{A}_n$ and $\mathcal{B}_n$ as

$$\mathcal{A}_n = -\frac{1}{c}\frac{\partial}{\partial t} h_n(z,t), \quad \mathcal{B}_n = \frac{\partial}{\partial z} h_n(z,t). \tag{12}$$

Study of (TM) modal fields is analogous. Solving Dirichlet boundary-eigenvalue problem for $\nabla_\perp^2$, that is,

$$\nabla_\perp^2 \phi_n + \kappa_n^2 \phi_n = 0, \quad \phi_n|_L = 0, \quad \frac{\kappa_n^2}{S}\int_S \phi_n^2\, ds = 1 \tag{13}$$

yields the vectorial elements of the waveguide modal basis as, $[\mathbf{z} \times \nabla_\perp \phi_n]$, $\nabla_\perp \phi_n$, and $[\mathbf{z}\, \kappa_n \phi_n]$, where $\phi_n \equiv \phi_n(\mathbf{r}_\perp)$. They compose the (TM) modal fields in the form of

$$\mathbb{H}_n^{\mathrm{TM}} = \mathfrak{A}_n(z,t)\, [\mathbf{z} \times \nabla_\perp \phi_n], \quad \mathbb{H}_{z\,n}^{\mathrm{TM}} \equiv 0 \tag{14a}$$

$$\mathbb{E}_n^{\mathrm{TM}} = \mathfrak{B}_n(z,t)\ \nabla_\perp \phi_n + e_n(z,t)\, [\mathbf{z}\, \kappa_n \phi_n]. \tag{14b}$$

Substation of these fields to Maxwell's equations result in KGE for $e_n(z,t)$ and direct formulas for $\mathfrak{A}_n$ and $\mathfrak{B}_n$ as

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} e_n(z,t) - \frac{\partial^2}{\partial z^2} e_n(z,t) + \kappa_n^2\, e_n(z,t) = 0 \tag{15a}$$

$$\mathfrak{A}_n = -\frac{1}{c}\frac{\partial}{\partial t} e_n(z,t), \quad \mathfrak{B}_n = \frac{\partial}{\partial z} e_n(z,t). \tag{15b}$$

It is worthwhile to notice that KGE (11) and (15a) carry all information about the contour L of the waveguide cross section. That is accumulated in the eigenvalues ν_n^2 and κ_n^2 for the TE- and TM-modes, respectively. If a shape of the contour L is composed of coordinate lines (Cartesian, cylindric or elliptic), the eigenvalues ν_n^2 and κ_n^2 can be obtained analytically in the process of solving the problems (9) and (13) by separation of the transverse waveguide coordinates. In these cases, the subscript $({}_n)$ is double. If a shape of the contour L is arbitrary, ν_n^2 and κ_n^2 can be obtained by solving (9) and (13) numerically and then substituted to appropriate KDE. It is proper to operate with the dimension-free variables as

$$\xi = \nu_n z, \quad \tau = \nu_n ct \quad \text{for TE-modes} \tag{16a}$$

$$\xi = \kappa_n z, \quad \tau = \kappa_n ct \quad \text{for TM-modes,} \tag{16b}$$

in terms of which the KGE assumes its canonical form as

$$\frac{\partial^2}{\partial\tau^2} f_n(\xi,\tau) - \frac{\partial^2}{\partial\xi^2} f_n(\xi,\tau) + f_n(\xi,\tau) = 0. \tag{17}$$

B. Oscillations in a Short-Circuited Piece of Waveguide

Consider the electromagnetic fields of the TE- and TM-types in a piece of the waveguide located between two cross sections at $z=0$ and $z=\ell$. Apply the boundary conditions of the perfect electric conductor at those cross sections as

$$h_n(z,t)|_{z=0,\ell} = 0 \quad \text{and} \quad \mathfrak{B}_n(z,t)|_{z=0,\ell} = 0, \tag{18}$$

see formulas (10a) – (10b) and (14a) – (14b). Apply separation of (z,t) variables in (11) as $f_n(z,t) = Z_n(z)\,T_n(t)$ what yields

$$\frac{1}{c^2}\frac{d^2}{dt^2}T_n(t)/T_n(t) + \nu_n^2 = \frac{d^2}{dz^2}Z_n(z)/Z_n(z) = -C \tag{19}$$

where C is a constant of separation of the variables. Boundary condition $h_n|_{z=0,\ell} = 0$ yields $C = (\pi m/\ell)^2$. Amplitude of the longitudinal component of the TE-modal cavity field is

$$h_{n\,m}(z,t) = \exp(i\,\omega_{n\,m}^{\mathrm{TE}} t)\sin(\pi m\,z/\ell) \tag{20}$$

where $m=1,2,\ldots$ and $\omega_{n\,m}^{\mathrm{TE}} = c\sqrt{\nu_n^2 + (\pi m/\ell)^2}$ is eigenfrequency of oscillations. Amplitudes of the transverse components of the TE-field in cavity can be calculated by formulas (12), what yields $\mathcal{A}_{n\,m} = -\frac{1}{c}\frac{\partial}{\partial t}h_{n\,m}$ and $\mathcal{B}_{n\,m} = \frac{\partial}{\partial z}h_{n\,m}$.

The same separating the variables in (15a) with applying the boundary condition $\mathfrak{B}_n|_{z=0,\ell} = 0$ results in the TM-modal cavity field oscillating with its eigenfrequency $\omega_{n\,m}^{\mathrm{TM}}$ as

$$e_{n\,m}(z,t) = \exp(i\,\omega_{n\,m}^{\mathrm{TM}} t)\cos[\pi m\,z/\ell] \tag{21a}$$

$$\mathfrak{A}_{n\,m} = -\frac{1}{c}\frac{\partial}{\partial t}e_{n\,m}(z,t), \quad \mathfrak{B}_{n\,m} = \frac{\partial}{\partial z}e_{n\,m}(z,t) \tag{21b}$$

where $m=1,2,\ldots$ and $\omega_{n\,m}^{\mathrm{TM}} = c\sqrt{\kappa_n^2 + [\pi m/\ell]^2}$.

C. Real-Valued Time-Harmonic Modal Waves

KGE supports propagation of the time-harmonic waves. Specifically, (17) has two linearly independent solutions, which correspond to the waves propagating along axis Oz. In the case of TE-wave, its longitudinal component is

$$\begin{aligned} h_n(\xi,\tau) &= a\sin(\varpi_n\tau - \beta_n\xi) + b\sin(\varpi_n\tau - \beta_n\xi) && (22a)\\ &= \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin\Psi + \frac{b}{\sqrt{a^2+b^2}}\cos\Psi\right] && (22b)\\ &= \sin(\theta + \varpi_n\tau - \beta_n\xi) && (22c)\end{aligned}$$

where $\varpi_n = \frac{\omega}{\nu_n c}$ is dimension-free frequency parameter, $-\infty<\omega<\infty$, $\beta_n = \sqrt{\varpi_n^2 - 1}$ is a propagation constant, $\Psi = (\varpi_n\tau - \beta_n\xi)$, a and b are arbitrary constants. As long as $(a/\sqrt{a^2+b^2})^2 + (b/\sqrt{a^2+b^2})^2 = 1$, we can denote $a/\sqrt{a^2+b^2} = \cos\theta$ and $b/\sqrt{a^2+b^2} = \sin\theta$ and substitute that in (22b). Combination of the trigonometric functions in (22b) yields the time-harmonic wave written in (22c) where θ is arbitrary constant, $0 \le \theta \le 2\pi$, and the first constant factor from (22b), $\sqrt{a^2+b^2}$, is ignored in (22c). Amplitudes of the transverse TE-wave components should be calculated by formulas (12) and substituted to (10a) – (10b).

Analogous manipulations with the longitudinal component of the TM-modal wave yield visually the same result as (22c):

$$e_n(\xi,\tau) = \sin\left(\vartheta + \hat{\varpi}_n\tau - \hat{\beta}_n\xi\right) \tag{23}$$

where $\hat{\varpi}_n = \frac{\omega}{\kappa_n c}$, $-\infty<\omega<\infty$, $\hat{\beta}_n = \sqrt{\hat{\varpi}_n^2 - 1}$ is a propagation constant, ϑ is arbitrary constant, $0 \le \vartheta \le 2\pi$. Definition (23) should be substituted to formulas (15a) – (15b) for transverse wave components and then they should be installed jointly to the vectorial fields (14a) – (14b). The cut-off frequencies of the TE- and TM-modal waves are specified by conditions $\beta_n = 0$ in (22c) and $\hat{\beta}_n = 0$ in (23), which result in

$$\omega_{n\,cut-off}^{\mathrm{TE}} = c\nu_n \quad \text{and} \quad \omega_{n\,cut-off}^{\mathrm{TM}} = c\kappa_n. \tag{24}$$

D. Miller's Orbits of Symmetry in Separation of Variables

KGE is a relativistic partial differential equation (PDE) of the hyperbolic type. Miller studied that one within the framework of *group theory* and discovered remarkable properties of its symmetry [6], [7]. He has deduced so-called *orbits of symmetry,* which have been turned out very useful for so-called incomplete separation of the variables (ξ,τ) in KGE. He proposed to consider the solution $f(\xi,\tau)$ to (25a) as a function with new variables, u and v, each of which is a function (unknown as yet) of the original variables, see (25b)

$$\begin{aligned}\left(\partial_\tau^2 - \partial_\xi^2 + 1\right) f(\xi,\tau) &= 0 && (25a)\\ f(\xi,\tau) &= f[u(\xi,\tau),\, v(\xi,\tau)] && (25b)\end{aligned}$$

where $\partial_\tau^2 \equiv \frac{\partial^2}{\partial\tau^2}$ and $\partial_\xi^2 \equiv \frac{\partial^2}{\partial\xi^2}$. Substituting (25b) to elegant equation (25a) changes its format drastically as

$$\begin{aligned}\Big\{&\underleftrightarrow{\left[(\partial_\tau u)^2 - (\partial_\xi u)^2\right]}\partial_u^2 + \underleftrightarrow{\left[(\partial_\tau v)^2 - (\partial_\xi v)^2\right]}\partial_v^2\\ &+ 2\underleftrightarrow{\left[(\partial_\tau v)(\partial_\tau u) - (\partial_\xi v)(\partial_\xi u)\right]}\partial_{u\,v}^2 && (26)\\ &+ \underleftrightarrow{\left[\partial_\tau^2 u - \partial_\xi^2 u\right]}\partial_u + \underleftrightarrow{\left[\partial_\tau^2 v - \partial_\xi^2 v\right]}\partial_v + 1\Big\} f(u,v) = 0\end{aligned}$$

So, now one has a problem: how to find such functions $u(\xi,\tau)$ and $v(\xi,\tau)$, which simplify equation (26) to a format that results in a *factorized* solution as $f(u,v) = U_\alpha(u)\,V_\alpha(v)$ where α is a constant of separation of the variables (u,v) in (26). Eventually, this problem has been solved by Miller when he specified the orbits of symmetry, which are listed below.

Orbit 1: $\tau = u$ and $\xi = v$, where $-\infty < u < \infty$, $-\infty < v < \infty$, yield a product $U(u)\,V(v)$ of the exponential functions by Miller. Implicitly, this (the lowest) orbit was used for development of the time-harmonic method since 1897. In Subsection D of this paper, studying the real-valued time-harmonic waves corresponds to Orbit 1.

Orbit 2: $\tau = u\cosh v$ and $\xi = u\sinh v$ with $0 \le u < \infty$, $-\infty < v < \infty$ yield product of an exponential and Bessel functions by Miller. The inverse to (τ,ξ) variables, which are applicable for substation to (26), are $u = \sqrt{\tau^2 - \xi^2}$ and $v = \frac{1}{2}\ln\frac{ct+z}{ct-z}$. The results were published in [8], [9], [10].

Orbit 3: $\tau = \left(u^2 + v^2\right)/2$ and $\xi = uv$ with $0 \le u < \infty$, $-\infty < v < \infty$ yield $f(u,v)$ as a product of parabolic cylinder functions by Miller. Inversion of (τ,ξ) variables yields

$u=\left(\sqrt{\tau+\xi}+\sqrt{\tau-\xi}\right)/\sqrt{2}$ and $v=\left(\sqrt{\tau+\xi}-\sqrt{\tau-\xi}\right)/\sqrt{2}$ provided that $\tau\geq\xi$. Results of studying published in [11] .

Orbit 4: $\tau=uv$ and $\xi=\left(u^2+v^2\right)/2$ with $0\leq u<\infty$, $-\infty<v<\infty$ yield product of parabolic cylinder functions by Miller. Inverse variables, which are applicable for substitution to (26), are $u=\left(\sqrt{\xi+\tau}+\sqrt{\xi-\tau}\right)/\sqrt{2}$ and $v=\left(\sqrt{\xi+\tau}-\sqrt{\xi-\tau}\right)/\sqrt{2}$. Causal fundamental solution was published in [12].

Orbit 5: $\tau+\xi=2\left(u+v\right)$ and $\tau-\xi=\left(u-v\right)^2$ with $-\infty<u,v<\infty$ yield product of Airy functions by Miller. Inverse variables applicable for substitution to (26), are $u=\left(\tau+\xi\right)/4-\sqrt{\tau-\xi}/2$ and $v=\left(\tau+\xi\right)/4+\sqrt{\tau-\xi}/2$. The solution was published in [13], [14].

Orbit 6: $\tau+\xi=\cosh\left[\left(u-v\right)/2\right]$ and $\tau-\xi=\sinh\left[\left(u+v\right)/2\right]$ with $-\infty<u,v<\infty$ yield product of Mathieu functions by Miller.

Orbit 7: $\tau+\xi=2\sinh\left(u-v\right)$ and $\tau-\xi=\exp\left(u+v\right)$ with $-\infty<u,v<\infty$ yield product of Bessel functions by Miller. Inverse variables applicable for substitution to (26) are $u=\ln(\sqrt{(\tau\text{-}\xi)\left[\tau+\xi+\sqrt{\left(\tau+\xi\right)^2+4}\right]/2})$ and $v=\ln(\sqrt{2\left(\tau\text{-}\xi\right)/\left[\tau+\xi+\sqrt{\left(\tau+\xi\right)^2+4}\right]})$ as the arguments of Bessel functions. Analysis of solution was given in [15].

Orbit 8: $\tau+\xi=2\cosh\left(u-v\right)$ and $\tau-\xi=\exp\left(u+v\right)$ with $-\infty<u,v<\infty$ yield product of Bessel functions by Miller.

Orbit 9: $\tau=\sinh u\cosh v$ and $\xi=\cosh u\sinh v$ with $-\infty<u,v<\infty$ yield product of Mathieu functions by Miller.

Orbit 10: $\tau=\cosh u\cosh v$ and $\xi=\sinh u\sinh v$ with $-\infty<u<\infty$, $0\leq v<\infty$ yield product of Mathieu functions by Miller.

Orbit 11: $\tau=\cos u\cos v$ and $\xi=\sin u\sin v$ with $0<u<2\pi$, $0\leq v<\pi$ yield product of Mathieu functions by Miller.

Orbits 6 and 8-11 are unstudied as yet. Besides, there are some other, but *non-orthogonal* systems of coordinates (u,v), which permit to separate the variables u and v in (26). They were listed in paper [16]. One can get information about the development of the EAE from publications [17]-[23], and from the other ones cited therein.

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