Electromagnetic inertia of the waveguide modes

(Invited Paper)

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***Abstract*—Studying electromagnetic inertia in SI units of the waveguide modes needs rearrangement of standard Maxwell's equations to a novel format where the electric and magnetic fields have common physical dimensions. The time-domain waveguide modes are found out explicitly via solving Maxwell's equations in their novel format in SI units within the framework of the evolutionary approach to electrodynamics. Electromagnetic mass and momentum of the waveguide modes are obtained as the mechanical equivalents of the energetic fields characteristics.**

I. Introduction

Studying inertia of the electromagnets fields in the *free space* was recently presented in [1] and in other publications cited herein. The authors derived criteria for the electromagnetic inertia under a strict requirement that the electric and magnetic fields are *summable.* In other words, both fields should have a *common* physical dimension. It is so if a researcher operates within the framework of the CGS metric system. In the generally accepted SI units, however, the electric and magnetic fields have *distinct* dimensions as *volt per meter,* $\lfloor \mathrm{V\,m^{-1}} \rfloor$, and *ampere per meter,* $\lfloor \mathrm{A\,m^{-1}} \rfloor$.

Our interest to the mechanical properties of the electromagnetic fields (and some other motivations) promoted us to reorganize the standard Maxwell's equations in SI units to a novel format (but staying in the SI metric system) where the electric and magnetic fields have common dimensions. The results in this direction were originally presented in [2], [3], and reported at this Symposium earlier [4]. In our version of Maxwell's equations all the electromagnetic quantities have their *common* dimension of *inverse meter* $\lfloor \mathrm{m^{-1}} \rfloor$ and involve only one *fundamental* physical constant: that is, the speed of light in vacuum, c, see equations (1a) – (1b) below.

Outline of this presentation involves three main parts, namely: **1.** Analytical solution of the boundary-value problem for studying propagation in waveguides for the time-varying waveforms (signals). **2.** Presentation of the modal energy density and the modal power flow density via new time-dependent vector fields, $\mathbb{E}$ and $\mathbb{H}$, which have their common dimension of *inverse meter* $\lfloor \mathrm{m^{-1}} \rfloor$. **3.** Derivation of mass and mechanical momentum as the mechanical equivalents of the energetic characteristics of the modal fields. The analytical results will be illustrated graphically.

A waveguide is geometrically homogenous along its axis Oz. Waveguide cross section, S, is bounded by a closed singly-connected contour, L. The contour is smooth enough in the sense that none of its possible *inner* angles (i.e., measured *within* S) exceeds π. In analysis what follows, a triplet of the mutually orthogonal unit vectors, $\mathbf{z}\times\mathbf{l} = \mathbf{n}$, will be used where unit vector $\mathbf{z}$ is axial, $\mathbf{l}$ and $\mathbf{n}$ are the tangential and normal to L unit vectors, respectively. The waveguide surface is perfectly conducting. To model possible losses, it is supposed that the waveguide is filled up with a rare electron gas for which Ohm's law holds as $\mathcal{J}=\sigma\mathcal{E}$ where σ is a constant of conductivity measurable in *siemens per meter,* $\lfloor \mathrm{S\,m^{-1}} \rfloor$.

So, equations (1a) – (1c) should be supplemented with the boundary conditions (1d). In total, the set of equations (1a) – (1d) composes standard statement of the boundary-value problem for the study of propagation in waveguides for the time-varying waveforms (signals) as follows:

$$\nabla\times\mathbb{H}(\mathbf{r},t) = \frac{1}{c}\frac{\partial}{\partial t}\mathbb{E}(\mathbf{r},t) + \gamma_0\,\mathbb{E}(\mathbf{r},t) \tag{1a}$$

$$\nabla\times\mathbb{E}(\mathbf{r},t) = -\frac{1}{c}\frac{\partial}{\partial t}\mathbb{H}(\mathbf{r},t) \tag{1b}$$

$$\nabla\cdot\mathbb{E}(\mathbf{r},t) = 0, \quad \nabla\cdot\mathbb{H}(\mathbf{r},t) = 0 \tag{1c}$$

$$\mathbf{l}\cdot\mathbb{E}|_L=0, \quad \mathbf{z}\cdot\mathbb{E}|_L=0, \quad \mathbf{n}\cdot\mathbb{H}|_L=0 \tag{1d}$$

where $\mathbb{E}$, $\mathbb{H}$, and γ_0 carry dimension of *inverse meter* $\lfloor \mathrm{m^{-1}} \rfloor$.

At any step of analysis, one can recover the standard electric and magnetic field vectors, $\mathcal{E}$ and $\mathcal{H}$, electric current density $\mathcal{J}$ by the scaling formulas given in [2]-[4] as

$$\mathcal{E}(\mathbf{r},t) = \epsilon_0^{\mathrm{V}}\mathbb{E}(\mathbf{r},t) \cong 3.3607\times10^5\,\mathbb{E}(\mathbf{r},t)\,\lfloor \mathrm{V\,m^{-1}} \rfloor \tag{2a}$$

$$\mathcal{H}(\mathbf{r},t) = \mu_0^{\mathrm{A}}\mathbb{H}(\mathbf{r},t) \cong 8.9206\times10^2\,\mathbb{H}(\mathbf{r},t)\,\lfloor \mathrm{A\,m^{-1}} \rfloor \tag{2b}$$

$$\mathcal{J}(\mathbf{r},t) = \mu_0^{\mathrm{A}}\mathbb{J}(\mathbf{r},t) \cong 8.9206\times10^2\,\mathbb{J}\,\lfloor \mathrm{A\,m^{-2}} \rfloor \tag{2c}$$

The scaling coefficients, ϵ_0^{V} and μ_0^{A}, were defined herein as

$$\epsilon_0^{\mathrm{V}} \overset{def.}{=} \sqrt{\mathrm{N}/\epsilon_0} \cong 3.3607\times10^5\,\lfloor \mathrm{V} \rfloor \tag{3a}$$

$$\mu_0^{\mathrm{A}} \overset{def.}{=} \sqrt{\mathrm{N}/\mu_0} \cong 8.9206\times10^2\,\lfloor \mathrm{A} \rfloor \tag{3b}$$

$$\gamma_0 \overset{def.}{=} \lfloor \mathrm{S\,m^{-1}} \rfloor\,\epsilon_0^{\mathrm{V}}/\mu_0^{\mathrm{A}} \cong \sigma_0\,376.73\,\lfloor \mathrm{m^{-1}} \rfloor \tag{3c}$$

where $\mathrm{N}\equiv\mathrm{kg\,m\,s^{-2}}$ is the force unit, *newton,* $\mathrm{V}\equiv\mathrm{N\,m}/(\mathrm{A\,s})$ is dimension of *volt,* A is *ampere,* $\lfloor \mathrm{S\,m^{-1}} \rfloor$ is dimension of *siemens per meter,* and number σ_0 is *quantity symbol* of σ (i.e., σ_0 is a *number* of units $\lfloor \mathrm{S\,m^{-1}} \rfloor$, which is present in σ given SI metric system). One can verify formulas (3a) – (3c) with using the definitions for ϵ_0 and μ_0 (see [5]), that is,

$$\epsilon_0 = \frac{1}{\mathrm{c}_0^2\,4\pi\times10^{-7}}\,\lfloor \mathrm{F\,m^{-1}}\equiv\mathrm{A^2\,s^2\,N^{-1}\,m^{-2}} \rfloor \tag{4a}$$

$$\mu_0 = 4\pi\times10^{-7}\,\lfloor \mathrm{H\,m^{-1}}\equiv\mathrm{N\,A^{-2}} \rfloor. \tag{4b}$$

where *number* c_0=2.99792458$\times10^8$ is the *quantity symbol* for the speed of light. For example, $c = 1/\sqrt{\epsilon_0\mu_0} = \mathrm{c}_0\,\lfloor \mathrm{m\,s^{-1}} \rfloor$.

II. Time-domain modes in the lossy waveguides

We have to solve problem (1a) – (1d) in the time domain. To this aim, the field vectors herewith should be sought as the *real-valued* vector functions of coordinates and time.

A. *Solving problem* (1*a*) – (1*d*) *for the* TE *modal fields*

Format of the transverse-electric (TE) waveguide modes is

$$\mathbb{E}^{\mathrm{TE}} = \mathcal{A}(z,t)\,\mathbf{e}(\mathbf{r}_\perp), \qquad \mathbb{E}_z^{\mathrm{TE}} \equiv 0 \tag{5a}$$

$$\mathbb{H}^{\mathrm{TE}} = \mathcal{B}(z,t)\,\mathbf{h}(\mathbf{r}_\perp) + h(z,t)\,\mathbf{h}_z(\mathbf{r}_\perp) \tag{5b}$$

where the vectorial factors should be find out as the elements of a modal basis in the waveguide cross-section domain S, $\mathbf{r}_\perp$ is two-component projection on S of the three-component position vector of a point of observation, $\mathbf{r} = \mathbf{r}_\perp + \mathbf{z}\,z$. The scalars, $\mathcal{A}$, $\mathcal{B}$, and h are the modal amplitudes. As a generator of the modal basis, we use a complete set of eigensolutions to the Neumann boundary-eigenvalue problem for the transverse part of Laplacian, $\nabla_\perp^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$, that is,

$$\left(\nabla_\perp^2 + \nu_n^2\right)\,\psi_n(\mathbf{r}_\perp) = 0, \quad \mathbf{n}\cdot\nabla_\perp\psi_n(\mathbf{r}_\perp)\,|_{\mathbf{r}_\perp\in L} = 0 \tag{6a}$$

$$\left(\nu_n^2/S\right)\int_S \psi_n^2 ds = 1 \tag{6b}$$

where $\nu_n^2 > 0$ are the eigenvalues and ψ_n are appropriate eigensolutions, subscript $(_n)$, $n = 1,2,\ldots$, organizes a distribution of ν_n^2 in increasing order of the numerical values on real axis. If the contour L is composed by the coordinate lines, the subscript n is double.

Example 1. A rectangle in Cartesian coordinates as $0 \le x \le a$, $0 \le y \le b$ is a typical waveguide cross-section. Solving Neumann problem (6a) with accuracy to a constant factor is

$$\psi_n(\mathbf{r}_\perp) \equiv \psi_{pq}(x,y) = \cos(\pi p x/a)\cos(\pi q y/b) \tag{7a}$$

$$\nu_n^2 \equiv \nu_{pq}^2 = (\pi p/a)^2 + (\pi q/b)^2 > 0 \tag{7b}$$

where $p = 0,1,2,\ldots$ and $q = 0,1,2,\ldots$ are integers.

The boundary conditions (1d) generate all elements of the modal basis for the field presentation (5a) – (5b) as

$$\mathbf{e}(\mathbf{r}_\perp) \equiv \mathbf{e}_n(\mathbf{r}_\perp) = \nabla_\perp\psi_n(\mathbf{r}_\perp)\times\mathbf{z} \tag{8a}$$

$$\mathbf{h}(\mathbf{r}_\perp) \equiv \mathbf{h}_n(\mathbf{r}_\perp) = \nabla_\perp\psi_n(\mathbf{r}_\perp) \tag{8b}$$

$$\mathbf{h}_z(\mathbf{r}_\perp) \equiv \mathbf{h}_{z\,n}(\mathbf{r}_\perp) = \mathbf{z}\,\nu_n\psi_n(\mathbf{r}_\perp) \tag{8c}$$

where $\psi_n(\mathbf{r}_\perp)$ is assumed as a dimension-free function. So, the complete set of TE-modes in format (5a) – (5b) is

$$\mathbb{E}_n^{\mathrm{TE}} = \mathcal{A}_n(z,t)\,\mathbf{e}_n(\mathbf{r}_\perp), \quad \mathbb{E}_{z\,n}^{\mathrm{TE}} \equiv 0 \tag{9a}$$

$$\mathbb{H}_n^{\mathrm{TE}} = \mathcal{B}_n(z,t)\,\mathbf{h}_n(\mathbf{r}_\perp) + h_n(z,t)\,\mathbf{h}_{z\,n}(\mathbf{r}_\perp) \tag{9b}$$

where the modal amplitudes are unknown as yet. It is worthwhile to notice that all the modal elements have dimension of $\lfloor \mathrm{m}^{-1} \rfloor$. Besides, the modal amplitudes, $\mathcal{A}_n$, $\mathcal{B}_n$, and h_n, should be found out as the dimension-free functions also.

A problem for the modal amplitudes can be obtained by substituting the fields (9a) – (9b) to Maxwell's equations (1a) – (1c) .The final result of these manipulations is

$$h_n(z,t) = \theta(z,t)\exp(-\tfrac{1}{2}\gamma_0 ct) \tag{10a}$$

$$\left(\tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2} - \tfrac{\partial^2}{\partial z^2} + \nu_n^2 - \gamma_0^2/4\right)\theta(z,t) = 0 \tag{10b}$$

$$\mathcal{A}_n = -\tfrac{1}{c}\tfrac{\partial}{\partial t}h_n(z,t), \quad \mathcal{B}_n = \tfrac{\partial}{\partial z}h_n(z,t) \tag{10c}$$

where Eq. (10b) is known as Klein-Gordon equation (KGE). The modal amplitudes $\mathcal{A}_n$ and $\mathcal{B}_n$ become dimension-free if one operates with the dimension-free variables specified as

$$\xi = \nu_n z \quad \text{and} \quad \tau = \nu_n ct \quad \text{for} \quad \text{TE-modes} \tag{11}$$

and the solutions to KGE, $\theta(\xi,\tau)$, are dimension-free also.

B. *Solving problem* (1*a*) – (1*d*) *for the* TM *modal fields*

To begin with, we shall define all elements of the modal basis for TM-modes in the waveguide cross-section domain S. For this purpose, the complete set of eigensolutions to Dirichlet boundary-value problem for the Laplacian, $\nabla_\perp^2$, will be used as a potential. Statement of that problem is

$$\left(\nabla_\perp^2 + \kappa_m^2\right)\,\phi_m(\mathbf{r}_\perp) = 0, \quad \phi_m(\mathbf{r}_\perp)\,|_{\mathbf{r}_\perp\in L} = 0 \tag{12a}$$

$$\left(\kappa_m^2/S\right)\int_S \phi_m^2 ds = 1 \tag{12b}$$

where $\kappa_m^2 > 0$ are the eigenvalues and ϕ_m are appropriate eigensolutions, subscript $(_m)$, $m = 1,2,\ldots$, organizes a distribution of κ_m^2 in increasing order of the numerical values on real axis. If the contour L is composed by the coordinate lines, the subscript m is double.

Example 2. Take again that rectangular domain S describable as $0 \le x \le a$, $0 \le y \le b$. Solving Dirichlet problem (12a) with accuracy to a constant factor is

$$\phi_m(\mathbf{r}_\perp) \equiv \phi_{pq}(x,y) = \sin(\pi p x/a)\sin(\pi q y/b) \tag{13a}$$

$$\kappa_m^2 \equiv \kappa_{pq}^2 = (\pi p/a)^2 + (\pi q/b)^2 > 0 \tag{13b}$$

where $p = 0,1,2,\ldots$ and $q = 0,1,2,\ldots$ are integers.

Boundary conditions (1d) suggest to define the basis elements in the cross-section domain, S, as follows:

$$\mathbf{h}_m(\mathbf{r}_\perp) = \mathbf{z}\times\nabla_\perp\phi_m(\mathbf{r}_\perp) \tag{14a}$$

$$\mathbf{e}_m(\mathbf{r}_\perp) = \nabla_\perp\phi_m(\mathbf{r}_\perp) \tag{14b}$$

$$\mathbf{e}_{z\,m}(\mathbf{r}_\perp) = \mathbf{z}\,\kappa_m\phi_m(\mathbf{r}_\perp) \tag{14c}$$

Hence, the TM-modal fields are presentable as

$$\mathbb{H}_m^{\mathrm{TM}} = \mathfrak{A}_m(z,t)\,\mathbf{h}_m(\mathbf{r}_\perp), \quad \mathbb{H}_{z\,m}^{\mathrm{TM}} \equiv 0 \tag{15a}$$

$$\mathbb{E}_m^{\mathrm{TM}} = \mathfrak{B}_m(z,t)\,\mathbf{e}_m(\mathbf{r}_\perp) + e_m(z,t)\,\mathbf{e}_{z\,m}(\mathbf{r}_\perp) \tag{15b}$$

with accuracy to the modal amplitudes of the field components. Substituting the fields (15a) – (15b) to Maxwell's equations (1a) – (1c) results in the problem for the amplitudes as

$$e_m(z,t) = \vartheta(z,t)\exp(-\tfrac{1}{2}\gamma_0 ct) \tag{16a}$$

$$\left(\tfrac{1}{c^2}\tfrac{\partial^2}{\partial t^2} - \tfrac{\partial^2}{\partial z^2} + \kappa_m^2 - \gamma_0^2/4\right)\vartheta(z,t) = 0 \tag{16b}$$

$$\mathfrak{A}_m = -\tfrac{1}{c}\tfrac{\partial}{\partial t}e_m(z,t), \quad \mathfrak{B}_m = \tfrac{\partial}{\partial z}e_m(z,t) \tag{16c}$$

The modal amplitudes $\mathfrak{A}_m$ and $\mathfrak{B}_m$ become dimension-free if one operates with the dimension-free variables specified as

$$\xi = \kappa_m z \quad \text{and} \quad \tau = \kappa_m ct \quad \text{for} \quad \text{TM-modes} \tag{17}$$

and the solutions $\vartheta(\xi,\tau)$ to KGE (16b) are dimension-free.

C. Real-valued time-harmonic modes in lossy waveguides

Problem (16a) – (16c) in terms of variables (17) looks as

$$e_m(\xi,\tau) = \vartheta(\xi,\tau)\exp(-\alpha_m\tau) \tag{18a}$$

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\xi^2} + 1 - \alpha_m^2\right)\vartheta(\xi,\tau) = 0 \tag{18b}$$

$$\mathfrak{A}_m = -\frac{\partial}{\partial\tau}e_m(\xi,\tau), \quad \mathfrak{B}_m = \frac{\partial}{\partial\xi}e_m(\xi,\tau) \tag{18c}$$

where $\alpha_m{=}\gamma_0/(2\kappa_m)<1$ is a dimension-free format of the lossy parameter γ_0 : see (1a). KGE (18b) supports propagation of the time-harmonic waves presentable as

$$\vartheta(\xi,\tau) = C\sin(\varphi + \varpi_m\tau - \beta_m\xi) \tag{19}$$

where $0 \le \varphi \le 2\pi$, $\varpi = \omega/(\kappa_m c)$ is dimension-free format of the frequency parameter ω, $-\infty < \omega < \infty$, and

$$\beta_m = \sqrt{\varpi_m^2 - (1{-}\alpha_m^2)} \tag{20}$$

is dimension-free propagation constant, C and φ are arbitrary constants. Presence of *two* free constants, C and φ, in (19) indicates that we take into account *two* linearly independent solutions to KGE (18b). Indeed, combination in (19) yields

$$\vartheta(\xi,\tau) = A\sin(\varpi_m\tau{-}\beta_m\xi) + B\cos(\varpi_m\tau{-}\beta_m\xi) \tag{21}$$

where $A{=}C\cos\varphi$ and $B{=}C\sin\varphi$. One can verify that the amplitudes in (18c) are dimension-free. Condition $\beta_m{=}0$ specifies the cut-off frequencies, $\omega_m^{cut-off}$, of the TM-modes, which propagate in the lossy waveguides, as

$$\omega_m^{cut-off} = \kappa_m c\sqrt{1 - (\gamma_0/2\kappa_m)^2} \tag{22}$$

where γ_0 is the lossy parameter, which was first introduced in (1a) and then specified in (3c).

D. Real-valued time-harmonic oscillations in a lossy cavity

A general cavity version of the EAE was given in [6], [7], [8]. Consider TE-oscillations in a short-circuited piece of the cylindrical waveguide located between two cross sections at $z=0$ and $z=\ell$. Apply the boundary condition of the perfect electric conductor at those cross sections as

$$h_n(z,t)\,|_{z=0,\ell} = \exp(-\tfrac{1}{2}\gamma_0 ct)\,\theta(z,t)\,|_{z=0,\ell} = 0, \tag{23}$$

see formulas (10a) – (10c) for the TE-modes (9a) – (9b). Apply factorization of the potential $\theta(z,t)$ as

$$\theta(z,t) = Z(z)\,T(t), \tag{24}$$

substitute (24) to (10b) and separate the variables what yields

$$\frac{1}{T(t)}\frac{d^2T(t)}{c^2dt^2} + \nu_n^2 - \gamma_0^2/4 = \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -\lambda^2 \tag{25}$$

where $-\lambda^2$ is a constant of separation of the variables. This yields a boundary-value problem for $Z(z)$ as

$$\frac{d^2}{dz^2}Z(z) + \lambda^2 Z(z) = 0, \quad Z(z)\,|_{z=0,\ell} = 0 \tag{26}$$

what results in

$$\lambda^2{\equiv}\lambda_s^2{=}(\pi\, s/\ell)^2, \quad Z(z){\equiv}Z_s(z){=}\sin\left(\pi s\tfrac{z}{\ell}\right) \tag{27}$$

where s is integer, The first part of equation (25) yields

$$\frac{d^2}{dt^2}T(t) + c^2\left(\nu_n^2{+}\left(\tfrac{\pi\, s}{\ell}\right)^2{-}\gamma_0^2/4\right)T(t) = 0. \tag{28}$$

This results in the time-harmonic solution as

$$T(t) \equiv T_{n\,s}(t) = \bar{C}\sin(\bar{\varphi} + \Omega_{ns}t) \tag{29}$$

where $\Omega_{ns}{=}c\sqrt{\nu_n^2{+}(\pi\, s/\ell)^2 {-}\gamma_0^2/4}$ is frequency of oscillations, $\bar{C}$ and $\bar{\varphi}$, $0 \le \bar{\varphi} \le 2\pi$, are arbitrary constants. Presence of these constants in (29) means that two linearly independent sine and cosine solutions of equation (28) are involved in the solution (29). In the case of the cavity with rectangular cross section (see **Example 1**) the frequency $\Omega_{ns} \equiv \Omega_{pqs}$ is

$$\Omega_{pqs}{=}c\sqrt{(\pi p/a)^2 + (\pi q/b)^2 + (\pi\, s/\ell)^2 - \gamma_0^2/4}\,. \tag{30}$$

Presence in (30) of the lossy parameter γ_0 (see (3c)) shows how γ_0 reduces the frequency of harmonic oscillations.

III. ENERGETIC CHARACTERISTICS OF THE MODAL FIELDS

Electromagnetic field energy density, $\mathcal{U}$, is specified via the standard field vectors, $\mathcal{E}$ and $\mathcal{H}$, as it is given in (31a). The quantity $\mathcal{U}$ carries dimension of *joule per meter*^{3} $\lfloor \mathrm{J\,m^{-3}} \rfloor$.

$$\mathcal{U}(\mathbf{r},t) = (1/2)\,(\epsilon_0\mathcal{E}\cdot\mathcal{E}{+}\mu_0\mathcal{H}\cdot\mathcal{H})\,\lfloor \mathrm{J\,m^{-3}} \rfloor \tag{31a}$$

$$= (1/2)\,(\,\epsilon_0\epsilon_0^{\mathrm{V}}\mathcal{E}\cdot\epsilon_0^{\mathrm{V}}\mathcal{E}{+}\mu_0\mu_0^{\mathrm{A}}\mathcal{H}\cdot\mu_0^{\mathrm{A}}\mathcal{H}\,) \tag{31b}$$

Substitutions of (3a)–(3b) to (31a) is shown in (31b). Simple manipulations result in the same quantity, but denoted as U since that is expressed via the field vectors, $\mathbb{E}$ and $\mathbb{H}$, as

$$U(\mathbf{r},t) = (1/2)\,(\,\mathbb{E}\cdot\mathbb{E} + \mathbb{H}\cdot\mathbb{H}\,)\,\lfloor \mathrm{N\,m^{-2}}{\equiv}\mathrm{J\,m^{-3}} \rfloor \tag{32}$$

where *joule* is product *newton meter*, $\lfloor \mathrm{J}{\equiv}\mathrm{N\,m} \rfloor$.

The power flow of the energy density is specified by Poynting vector, $\mathcal{S}$, expressed via the fields $\mathcal{E}$ and $\mathcal{H}$ as

$$\mathcal{S}(\mathbf{r},t) = [\mathcal{E}\times\mathcal{H}] = [\,\epsilon_0^{\mathrm{V}}\mathcal{E}\times\mu_0^{\mathrm{A}}\mathcal{H}\,]\,\lfloor \mathrm{W\,m^{-2}} \rfloor \tag{33}$$

where *watt*, $\lfloor \mathrm{W} \rfloor$, is *joule per second*, $\lfloor \mathrm{W}{=}\mathrm{J\,s^{-1}} \rfloor$. The same power flow, but expressed via the fields $\mathbb{E}$ and $\mathbb{H}$, is

$$\mathbb{S}(\mathbf{r},t) = c\,[\mathbb{E}\times\mathbb{H}]\,\lfloor \mathrm{N\,m^{-1}\,s^{-1}}{\equiv}\mathrm{W\,m^{-2}} \rfloor \tag{34}$$

where $c = 1/\sqrt{\epsilon_0\mu_0} = \mathrm{c}_0\,\lfloor \mathrm{m\,s^{-1}} \rfloor$ is the speed of light.

Umov had proved that velocity of transportation of energy in *any* wave process is a simple fraction where the power flow stands in numerator and stored energy in denominator [9]. According to Poynting's theorem [10], velocity of transportation of the modal field energy is written below in terms of the standard field vectors and in term of the new ones as

$$\mathcal{V}(\mathbf{r},t) = \frac{\mathcal{S}(\mathbf{r},t)}{\mathcal{U}(\mathbf{r},t)} = \frac{2[\mathcal{E}\times\mathcal{H}]}{\epsilon_0\mathcal{E}\cdot\mathcal{E}+\mu_0\mathcal{H}\cdot\mathcal{H}}\,\lfloor \mathrm{m\,s^{-1}} \rfloor \tag{35a}$$

$$\mathbb{V}(\mathbf{r},t) = \frac{\mathbb{S}(\mathbf{r},t)}{U(\mathbf{r},t)} = c\,\frac{2\,[\mathbb{E}\times\mathbb{H}]}{\mathbb{E}\cdot\mathbb{E}+\mathbb{H}\cdot\mathbb{H}}\,\lfloor \mathrm{m\,s^{-1}} \rfloor\,. \tag{35b}$$

IV. Inertial properties of the modal fields

Scaling the vector $\mathbb{S}$ by the light velocity, c, yields

$$\dddot{\mathbb{S}}(\mathbf{r},t) = \mathbb{S}(\mathbf{r},t)/c = [\mathbb{E}\times\mathbb{H}] \left\lfloor \mathrm{N\,m^{-2}} \equiv \mathrm{J\,m^{-3}} \right\rfloor . \tag{36}$$

Notice that the new *vector,* $\dddot{\mathbb{S}}$, and the *scalar* U (see (34)) have common their dimensions. Following Kaiser's technique [1], compose a pair of new *scalars (!)* as follows:

$$\mathfrak{U}(\mathbf{r},t) = \sqrt{\tfrac{1}{4}\left(\mathbb{E}^2+\mathbb{H}^2\right)^2} = \sqrt{\tfrac{1}{4}\left(\mathbb{E}^4+2\mathbb{E}^2\mathbb{H}^2+\mathbb{H}^4\right)} \tag{37a}$$

$$\mathfrak{I}(\mathbf{r},t) = \sqrt{(\mathbb{E}\times\mathbb{H})\cdot(\mathbb{E}\times\mathbb{H})} = \sqrt{\mathbb{E}^2\mathbb{H}^2-(\mathbb{E}\cdot\mathbb{H})^2} \tag{37b}$$

where dot product $[\mathbb{E}\times\mathbb{H}]\cdot[\mathbb{E}\times\mathbb{H}]$ is found out by applying identity from [[11], eq. (B.8)]. Combination of $\mathfrak{U}$ and $\mathfrak{I}$ as

$$\mathrm{R}(\mathbf{r},t) = \sqrt{\mathfrak{U}^2-\mathfrak{I}^2} = \tfrac{1}{2}\sqrt{\left(\mathbb{E}^2-\mathbb{H}^2\right)^2+4(\mathbb{E}\cdot\mathbb{H})^2}\,\left\lfloor \mathrm{J/\,m^3} \right\rfloor \tag{38}$$

was defined as the *reactive (rest) energy density* in [1].

In relativistic mechanics, the general relationship between energy E of a particle with its mass m and momentum p in the state of rest is as follows:

$$\mathrm{E} = c\sqrt{\mathrm{p}^2+\mathrm{m}^2c^2} \quad \text{and} \quad \mathrm{p} = \mathrm{m}\mathbf{v}/\sqrt{1-\mathbf{v}^2/c^2} \tag{39}$$

where $\mathbf{v}$ is velocity of the particle in the chosen inertial reference frame [12]. We shall use this statement for derivation of the inertial properties of the electromagnetic modal waves. First notice that in the reference frame of rest, where $\mathbf{v}=\mathbf{0}$, equations (39) result in famous Einstein's formula $\mathrm{E}=\mathrm{m}c^2$. And vice versa, the mass m of the particle (as a measure of its inertia) is expressible via its energy as

$$\mathrm{m} = \mathrm{E}/c^2. \tag{40}$$

By analogy, electromagnetic inertia (in the reference frame of rest) can be specified via replacing the energy E in (40) by the electromagnetic energy R given in (38) what yields

$$\mathrm{m_{EM}} = \mathrm{R}(\mathbf{r},t)/c^2 \tag{41a}$$

$$= \frac{1}{2\,\mathrm{c}_0^2}\sqrt{\left(\mathbb{E}^2-\mathbb{H}^2\right)^2+4(\mathbb{E}\cdot\mathbb{H})^2}\,\left\lfloor \mathrm{kg/\,m^3} \right\rfloor \tag{41b}$$

At this point, two important comments should be made: **(1)** the quantities $\mathbb{E}^2-\mathbb{H}^2$ and $\mathbb{E}\cdot\mathbb{H}$ are the invariants with respect to the inertial reference frames, and **(2)** the fields $\mathbb{E}$ and $\mathbb{H}$ are orthogonal in all the waveguide modes and hence, $(\mathbb{E}\cdot\mathbb{H})=0$. Thus, we have for the reference frame of rest as

$$\mathrm{m_{EM}}c^2 = \mathrm{R}(\mathbf{r},t) = \tfrac{1}{2\,\mathrm{c}_0^2}\left|\mathbb{E}^2-\mathbb{H}^2\right| \left\lfloor \mathrm{J/\,m^3} \right\rfloor \tag{42}$$

accordingly to (41a) , (38) and that condition $(\mathbb{E}\cdot\mathbb{H})=0$. When the particle energy E is large compared to its rest energy $\mathrm{m}c^2$, Landau specifies momentum as $\mathrm{p}=\mathrm{E}/c$ (see [12] eq. (9.9)). We replace therein E by $\mathfrak{I}$ (see (37b)) what yields definition (43a) for the momentum of modal fields as

$$\mathrm{p} = \mathrm{E}/c = \mathfrak{I}/c \tag{43a}$$

$$\mathrm{p_{EM}} = \mathfrak{I}/\mathfrak{c} = \tfrac{1}{\mathrm{c}_0}\,|\mathbb{E}|\;|\mathbb{H}|\,\left\lfloor \mathrm{kg\,\tfrac{m}{s}/\,m^3} \right\rfloor . \tag{43b}$$

In (43b) , calculations are presented for volumetric distribution of a *mechanical* momentum of the waveguide modes.

Graphical results are exhibited below provided that the dot products of the vectors (i.e., $\mathbb{E}^2$ and $\mathbb{H}^2$) are averaged over the waveguide cross section S. For the TE-modal fields, the results are

$$\tfrac{1}{S}\int_S \mathbb{E}_n^{\mathrm{TE}}\cdot\mathbb{E}_n^{\mathrm{TE}}ds = \mathcal{A}_n^2(z,t) \tag{44a}$$

$$\tfrac{1}{S}\int_S \mathbb{H}_n^{\mathrm{TE}}\cdot\mathbb{H}_n^{\mathrm{TE}}ds = \mathcal{B}_n^2(z,t)+h_n^2(z,t) \tag{44b}$$

where normalization condition (6b) in calculations. The averaged TM-modal fields are

$$\tfrac{1}{S}\int_S \mathbb{H}_m^{\mathrm{TM}}\cdot\mathbb{H}_m^{\mathrm{TM}}ds = \mathfrak{A}_m^2(z,t) \tag{45a}$$

$$\tfrac{1}{S}\int_S \mathbb{E}_m^{\mathrm{TM}}\cdot\mathbb{E}_m^{\mathrm{TM}}ds = \mathfrak{B}_m^2(z,t)+e_m^2(z,t) \tag{45b}$$

where normalization condition (12b) was used.

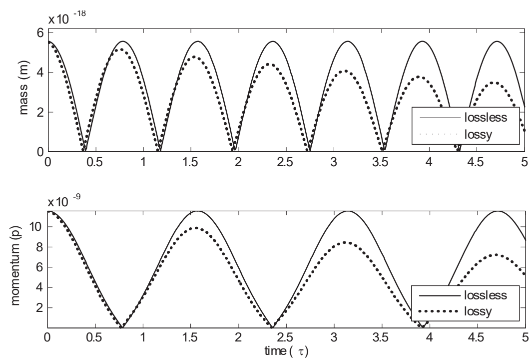

Fig. Electromagnetic inertia of the waveguide modes: dependence on time of mass (above) and momentum (below).

Notice that $\mathrm{p_{EM}}$ in (43b) is absolute value of the momentum. We imply that *vector* $\tilde{\mathrm{p}}_{\mathrm{EM}}$ is colinear to Poynting vector.

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