Gaussian Beams in Orthogonal Full Ray Trajectory Coordinates

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Abstract

Special type of the Gaussian beam is constructed in the coordinate system of the full ray trajectories. The attention is paid on the effect of beam "reflecting" from a caustic. To describe such effect, this requires developing the special procedure of regularization of the appropriate integral, which is singular due to the zero-value of the cross section of ray tube at a caustic.

1. Introduction

In mathematical physics the special type waveform as the Gaussian beam is widely studied as one of the solutions to the parabolic equation, or the complex valued solutions to the equations of the Geometrical Optics (GO) approximation (see, e.g., [1-7]; and many others). When the medium of propagation is homogeneous, the appropriate equations are straightforward. In the case of propagation in the inhomogeneous medium it may be convenient to introduce the curvilinear coordinate system in order to deal with the problem under consideration.

Depending on the curvilinear coordinates introduced, relevant Lame coefficients may not become equal zero, as in the case of the local reference ray-centered coordinate system [1, 3, 5], or may become zero at some points along the paths of propagation. The latter occurs in the case, where the orthogonal coordinates of the full ray path trajectories are introduced [2, 4, 7] with the appropriate Lame coefficient being the divergence of a ray tube (bundle). This coefficient becomes equal zero at the point, where a ray path touches a (simple) caustic. In turn, this results in the singular point of the appropriate differential equation for the field, and formally leads to loss of the uniqueness of the solution behind this point.

2. Statement of the Problem

The described above case of the singular Lame coefficient may occur, in particular, for the HF propagation in the ionospheric reflection channel, when calculating the field propagated along a given path started from a point on the Earth's surface and arriving at another point on the Earth's surface after reflecting from a simple caustic in the ionosphere (see the sketch in Fig. 1, taken from [8]).

When calculating the HF field amplitude along any ray path touching the caustic in the GO approximation employing ray-tracing, one faces the problem of singularity of the amplitude on caustic, so that

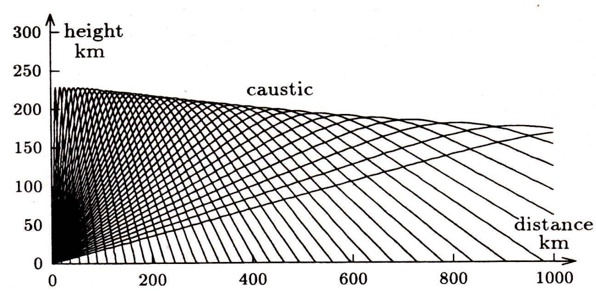

Figure 1. Ray paths in the parabolic ionospheric layer emanated from the point source located at the origin of the coordinate system (approximation of the plane Earth, 2-D case) for the monochromatic signal with the transmission frequency smaller, than the maximal plasma frequency in the ionosphere.

the procedure should be introduced in order to continue the field amplitude behind the point, where the path touches the appropriate caustic. To solve this in the GO approximation, the ray path equations are supplemented by the equations for the derivatives of the ray trajectory in the initial conditions (in the case of the full ray trajectory coordinates, associated with the point source of the field, these are the derivatives of the trajectory in the angles of emanation of the ray paths from the source of the field). It appears so that the solutions to these equations are not sensitive to (are not singular in) the special point, where the ray trajectory meets the caustic. Along with this, when constructing the complex GO field amplitude, the appropriate determinant, which defines the ray tube divergence and stands in the denominator of the GO amplitude, becomes equal to zero. As the result, all the outlined permits passing through the singular point and uniquely and properly extend the GO field amplitude along a given ray path to the point after touching a caustic. In the same style, the problem can be formulated how to extend the Gaussian beam, generated in the full ray trajectory coordinates around any specified ray path in the vicinity of the point source (see Fig. 1), to the area behind the point, where the chosen ray path touches caustic. This problem will be discussed below for the scalar case on the basis of the parabolic equation written for the correspondent Helmholtz equation in the two-dimension problem.

3. Main Relationships

The scalar harmonic wave field, radiated by the point source is described by the following scalar equation:

$$\nabla^2 E + k^2 \varepsilon(x,z) E = \delta(x)\delta(z). \qquad (1)$$

Here $\varepsilon(x,z)$ is the dielectric permittivity; k is the wave number in vacuum.

The full ray trajectory orthogonal coordinates (τ, ϑ) are introduced in equation (1), where τ is the eikonal (full phase path) along the curved path of propagation, and ϑ is the angle between the *z*-axis of the rectangular coordinates and the direction of the ray path emanating from the point source, placed at the origin of the coordinate system. For these coordinates the appropriate Lame coefficients are

$$h_\tau(\tau,\vartheta) = \varepsilon^{-1/2}(\tau,\vartheta), \quad h_\tau(\tau,\vartheta) = a(\tau,\vartheta), \qquad (2)$$

where $a(\tau,\vartheta)$ is the cross section of the ray tube (bundle). When writing equation (1) utilizing the introduced orthogonal curvilinear coordinate system and performing the substitution

$$E(\tau,\vartheta) = a^{-1/2}(\tau,\vartheta)\, \varepsilon^{-1/4}(\tau,\vartheta) e^{ik\tau} U(\tau,\vartheta), \qquad (3)$$

the following parabolic equation for the new unknown function $U(\tau,\vartheta)$ can be derived as follows:

$$2ik\varepsilon(\tau,\vartheta)\frac{\partial U}{\partial \tau} + a^{-2}(\tau,\vartheta)\frac{\partial^2 U}{\partial \vartheta^2} = 0 \,. \qquad (4)$$

The Gaussian beam type solution to equation (3), localized around a specified ray path $\vartheta = \vartheta_0$, can be found in the form:

$$U(\tau,\vartheta) = F(\tau) exp[ig(\tau)(\vartheta - \vartheta_0)^2]. \qquad (5)$$

In the relationship (5) function $g(\tau)$ is the explicit complex valued solution to the appropriate Riccati equation represented as

$$g(\tau) = \frac{1}{2}\left[\int_0^\tau \frac{d\tau\prime}{k\varepsilon(\tau\prime,\vartheta_0)a^2(\tau\prime,\vartheta_0)} - \frac{1}{id}\right]^{-1}. \qquad (6)$$

Here d is the real constant, which sign is chosen in order to provide the field localization around any specified ray path ϑ_0.

Function $F(\tau)$ is expressed through the function $g(\tau)$ in the form as follows:

$$F(\tau) = F(0)\left(\frac{g(\tau)}{g(0)}\right)^{1/2}. \qquad (7)$$

Relationships (3, 5 - 7) give the Gaussian beam type solution to equation (4) in the vicinity of an arbitrary specified by ϑ_0 path, built in the full ray trajectory coordinate system. When analyzing these, it should be taken into account that in the vicinity of the point (τ_0, ϑ_0), where this path touches caustic, function $a(\tau,\vartheta_0)$ behaves as the linear function of $(\tau - \tau_0)$. In turn, this means that the integrals (6, 7) are the singular integrals, once the point of observation τ along the specified by $\vartheta_0$ray path approaches caustic and then goes behind the point τ_0. However, taking account of all the factors in (3) and (5) before the exponential function in (3), all these together produce a non-singular product, so that the only singular problem is with the integral (7). The matter is that this integral does not converge even in the sense of its principle value, and the special procedure was elaborated for its regularization.

In the most general case of full 2-D inhomogeneous medium calculations according to the relationships presented above can solely be performed numerically. However, in the case of a plane stratified medium ($\varepsilon(x,z) = \varepsilon(z)$) some analytical results are available. In particular, the analytical expression for the coefficient Lame $h_\tau(\tau,\vartheta) = a(\tau,\vartheta)$, can be obtained employing necessary relationships in [8, Chapter 5, pp. 65-68]. In the next Section the results of numerical calculations for the Gaussian beam, propagating in the plane stratified 2-D ionospheric layer and "reflected" from the caustic as in Fig. 1, will be presented.

4. Results

Calculations were performed for the Chapman model layer of the vertical electron density profile employing the theory presented above. The parameters of the layer are the following: the height of the maximum of the layer, h_m= 300 km; the characteristic scale H= 60 km; the maximal plasma frequency in the layer $f_{p\,max}$= 10 MHz, the transmission frequency f= 6 MHz; the ray path starts from the origin of the coordinate system at the angle of 40^0 with the normal to the Earth's surface.

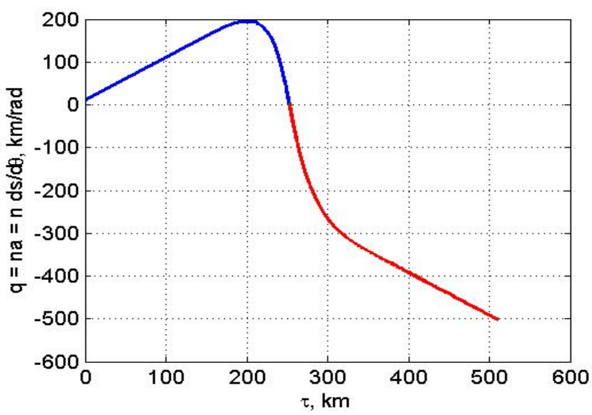

Figure 2. Function $q(\tau,\vartheta_0)$ as the function of variable τ along the specified ray path.

In Fig. 2 the dependence of the quantity $q(\tau, \vartheta_0) = \sqrt{\varepsilon(\tau, \vartheta_0)}a(\tau, \vartheta_0)$ as the function of the eikonal τ along the path of propagation is shown. The sign of this function is changed at τ_0.

The following figures below illustrate the behavior of the Gaussian beam when propagating along the described above path of propagation, formed at the distance of 10 km from the point source, which is placed at the origin of the coordinate system, and having the initial angle width of 0.1 *rad*. This is specified by the value of the quantity $g(0) = id$ (d is pure imaginary).

In Fig. 3 the real (blue) and imaginary (green) parts of the function $g(\tau)$, defined in eq. (6), are shown as the function of the eikonal along the ray path. The zero value of $g(\tau)$ (both real and imaginary parts are equal to zero) corresponds to the point τ_0, where the path touches caustic.

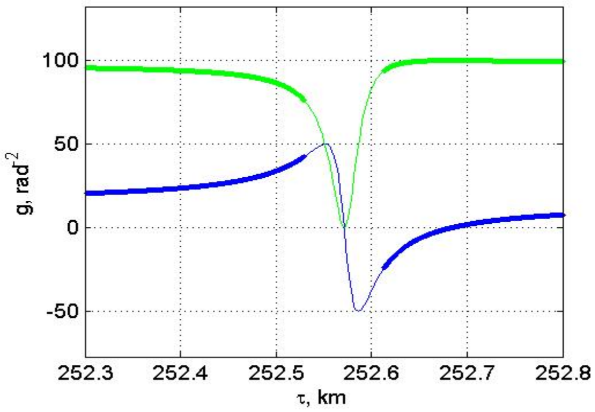

Figure 3. Real (blue line) and imaginary (green line) parts of the function $g(\tau)$.

According to Fig. 3, a local change of the front curvature (blue curve) takes place near caustic in the style that at caustic the additional curvature becomes equal to zero, so that the full curvature of the front at this point is fully defined by the factor $exp(ik\tau)$ in equation (3).

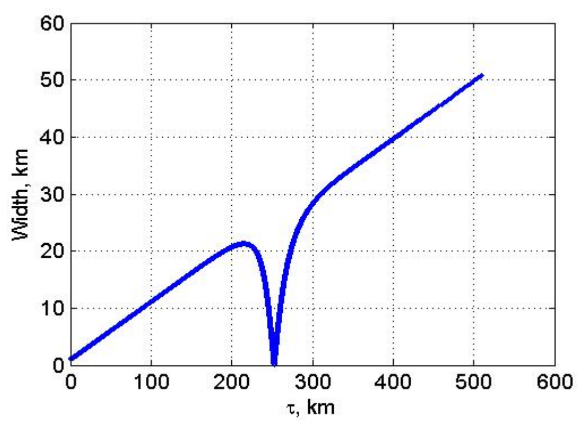

Figure 4. Change of the beam width along the ray path. Here the minimal width of the beam corresponds to the point behind the caustic.

The next Figs 4 and 5 show the evolution of the beam width observed along the specified by ϑ_0 ray path in more detail.

In the Fig. 5 below the fragment of the curve from the previous figure, containing the minimum, is shown in a finer scale

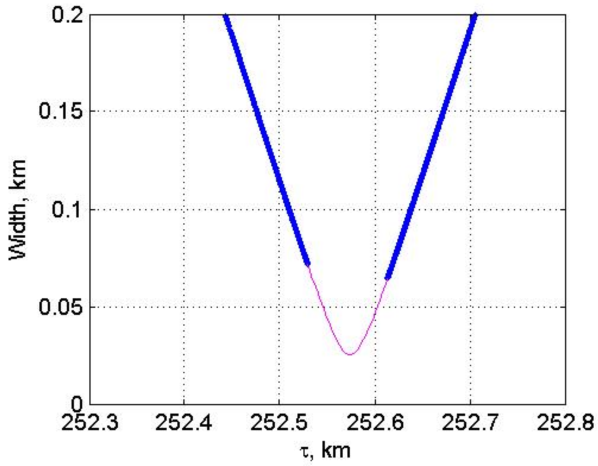

Figure 5. Fragment of the line in previous Fig. 4. The thin part of the line indicates the area in the vicinity of the point, where the ray path touches caustic.

Finally, the behavior of the modulus and phase of the complex amplitude (pre-exponential factor in equation (3)) is shown in Fig. 6. As is seen, its phase changes fast in the area, where the absolute value has its maximum (in the area of focusing of the beam).

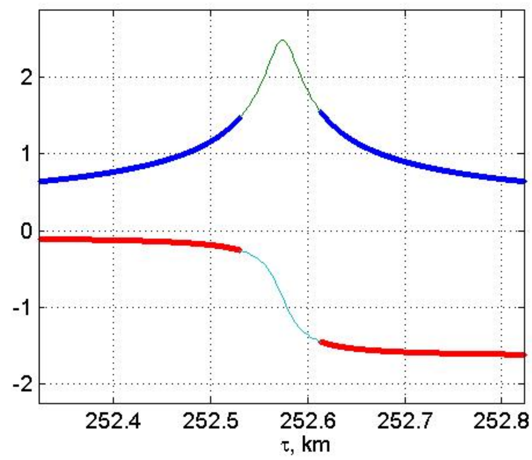

Figure 6. Absolute value (blue) and phase (red) of the complex amplitude in equation (3).

In conclusion, the localized solution of the Gaussian beam type was constructed for the 2-D inhomogeneous medium of propagation, where the full ray trajectory coordinate system was introduced. The special consideration was given to the case, when the ray path goes through the point, where the appropriate Lame coefficient becomes equal to zero (touches a caustic). To consider this case the special procedure of the appropriate singular integral regularization was required.

5. Acknowledgement

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6. References

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