2nd URSI AT-RASC, Gran Canaria, 28 May – 1 June 2018

Multifractal Phase Screen Model for Scintillation of Transionospheric Signals

Makarenkova E.V.[1*] and Gherm V.E.[1]
(1) St. Petersburg State University, St. Petersburg, 199034, RUSSIA,. st023937@spbu.ru, v.germ@spbu.ru

Abstract

This work deals with the problem of electromagnetic field propagation in the turbulent ionosphere with intermittency effects taken into account. The intermittency is involved through introducing the multifractal stochastic distribution of irregularities that implies the non-Gaussian statistics of the electron density fluctuations. The propagation problem is simulated utilizing the random screen technique. The algorithm of generation of the spatial distribution of the stochastic phase on the screen employs the two-dimensional inverse discrete wavelet transform. The wavelet coefficients are generated utilizing the stochastic lognormal cascade on the dyadic tree. The statistical characteristics of the field fluctuations after passing the phase screen are studied, and the peculiarities of the field statistics due to the non-Gaussianity of the screen are demonstrated and discussed.

1. Introduction

The fluctuating transionospheric signals are typically intermittent, i.e. their phase and amplitude often demonstrate a non-homogeneous, spiky behavior. Since the fluctuations of the signal arise due to the scattering on the irregularities of the ionospheric electron density, it is natural to suppose that the intermittency of scintillation is a consequence of the intermittent character of turbulent fluctuations of the electron density in the ionosphere. The intermittency is inherent to the turbulence phenomena, manifesting itself as strong irregular local deviations of the parameters of the turbulent medium (density, velocity, concentration etc.) from their mean values. In other words, large fluctuations appear to be more probable than one would expect from normal distributions. The occurrence of such strong deviations leads to the non-Gaussianity of the fluctuations of parameters. Such a non-Gaussian behavior is experimentally observed in satellite and rocket in-situ measurements of the ionospheric irregularities [1].

In the majority of the existing theories of scintillation of the transionospheric signals, the employed models of the ionospheric electron density fluctuations are typically introduced as stochastic distributions with Gaussian probability density function (PDF) and given correlation or structure function or, equivalently, with given spectra of irregularities. In so doing, the description of irregularities is based on the first two statistical moments of fluctuations (mean value and autocorrelation function), which gives the full statistical characterization in case of Gaussian random fields, but is not sufficient to provide the description of the intermittent features of the electron density irregularities. As for the spatial spectra of irregularities, at present time it is commonly accepted and supported by the numerous experimental evidences (see, e.g., [2]) that it can be modeled by an inverse power low dependence of spectral density of fluctuations on the wave number. The corresponding structure function of the electron density fluctuations is also a power type function of spatial separation. For instance, for the isotropic three-dimensional Kolmogorov turbulence, the structure function of the chaotic velocity is proportional to the separation distance to the power of 2/3, while the corresponding spectrum decays as the wave number to the power of -11/3. The random functions obeying the power low scaling may be regarded as stochastic fractals, which can be characterized by the single power law exponent – Hurst exponent.

The modern theories of turbulence take account of the intermittency or non-homogeneity of the distribution of parameters in space, introducing the conception of the multifractals, or non-homogeneous fractals, which are characterized by a whole spectrum of the local power law exponents, or by the multifractal spectrum [3].

The problem of wave propagation in a non-Gaussian fluctuating medium was considered only in a restricted number of publications. Thus, Boldyrev and Gwinn [4] considered radio waves propagating through the interstellar medium assuming the stable distribution of electron density fluctuations of the Lévy flight type statistics. Wernik and Grzesiak [1] numerically modeled transionospheric propagation utilizing the Laplace (double exponential) PDF of the electron density irregularities employing the multiple two-dimensional phase screen method.

This work deals with the problem of the field propagation in the turbulent ionosphere with intermittency effects taken into account. The intermittency is involved through the introducing the multifractal stochastic distribution of irregularities which implies the non-Gaussian statistics of electron density fluctuations.

2. The non-Gaussian Random Phase Screen

In order to produce the first simplest estimates of the impact of intermittent fluctuations of the electron density on the scattered field, the most simple propagation model is employed as the first step, which is the stochastic phase screen. Historically, the phase screen model was the first one employed in the problems of wave propagation in the random media. The theory of the phase screen is fully elaborated in a huge number of papers and books for the case of Gaussian random phase distributions. The theory of the non-Gaussian stochastic phase screen is not so well developed; in this respect, it is worth mentioning the book [5], where some non-Gaussian models, in particular, the generalized gamma process and random telegraph wave have been considered.

This paper introduces the general non-Gaussian multifractal model for the stochastic distribution of phase, which is characterized by the generalized structure functions of different orders

$$S_q(L) = \langle |\varphi(r+L) - \varphi(r)|^q \rangle , \tag{1}$$

here φ is the phase, L is the spatial separation and q is the order of the structure function. The angle brackets denote the statistical averaging over the ensemble of realizations of the random spatial phase distributions. The case $q = 2$ corresponds to the ordinary structure function of the second order. For Gaussian fractals the phase differences for each separation are the normally distributed random values, therefore one can obtain

$$S_q(L) \sim L^{Hq} , \tag{2}$$

where H is the Hurst exponent. For the one-dimensional fractal functions the Hurst exponent is related to the fractal dimension D as $D = 2 - H$. It is seen from (2) that the Gaussian fractals are characterized by the single Hurst exponent, and the structure functions of higher orders can be directly determined if the structure function of the order 2 is known. For the non-Gaussian fractals, or multifractals, the structure functions take the form as follows

$$S_q(L) \sim L^{\zeta(q)} , \tag{3}$$

where the scaling exponents $\zeta(q)$ is no longer the linear function of q ; then in some interval of q it can be approximated by a quadratic dependence which is characteristic for the multifractal random functions

$$\zeta(q) = hq - \frac{\lambda^2 q^2}{2} . \tag{4}$$

In the representation (4), the two parameters, h and λ^2 (instead of the only one Hurst exponent in (2)) determine the scaling properties of the multifractal distribution. Apparently, the departure of scaling exponents from the linear law and, hence, the deviation of the random process from Gaussianity can be characterized by the parameter λ^2, which we will refer to as the intermittency parameter.

To generate the stochastical distribution of phase on the screen obeying the scaling properties (3, 4), the cascade model is employed which mimics the energy transfer from large scales to the smaller ones. The model has been developed in [6] and allows generating the stochastic distribution with the prescribed parameters of the multifractal spectrum. This distribution after the appropriate normalization will be utilized as the distribution of phase on the screen.

The algorithm of generation employs the two-dimensional inverse discrete wavelet transform. The two-dimensional stochastic distribution of the phase is specified as a 2D wavelet decomposition

$$\varphi(x,y) = c_N \phi_N(x,y) + \sum_{\alpha=1}^{3} \sum_{j=0}^{N} \sum_{m,n=0}^{2^{N-j}-1} c_{j,m,n}^{\alpha}\, \psi_{j,m,n}^{\alpha}(x,y), \tag{5}$$

where $\{\phi(x,y),\ \psi_{j,m,n}^1(x,y),\ \psi_{j,m,n}^2(x,y),\ \psi_{j,m,n}^3(x,y)\}$ is the compactly supported orthonormal 2D wavelet basis, comprised of the horizontal, vertical and diagonal 2D wavelets. The wavelet coefficients $c_{j,m,n}^{\alpha} = \langle \psi_{j,m,n}^{\alpha} | \varphi \rangle$ are generated employing a multiplicative cascade algorithm on a dyadic grid [7].

For each random realization of a distribution, the wavelet coefficients are generated in two steps. Firstly, the coefficients on the largest scale ($c_N,\ c_{0,0,0}^{\alpha}$) are specified. Then the cascade of d-coefficients is recursively built in the following way:

$$d_{0,0,0} = \sqrt{\left[c_{0,0,0}^1\right]^2 + \left[c_{0,0,0}^2\right]^2 + \left[c_{0,0,0}^3\right]^2} \tag{6}$$

$$d_{j,m,n} = W_{j-1,m,n}^{(r1)} \cdot d_{j-1,m,n} \qquad d_{j,m,n+1} = W_{j-1,m,n}^{(l1)} \cdot d_{j-1,m,n}$$
$$d_{j,m+1,n} = W_{j-1,m,n}^{(r2)} \cdot d_{j-1,m,n} \qquad d_{j,m+1,n+1} = W_{j-1,m,n}^{(l2)} \cdot d_{j-1,m,n}$$

where j is the number of the level of decomposition:

$$1 \le j \le N, \quad 0 \le m,n \le 2^{N-j} - 1 . \tag{7}$$

The multipliers W are the independent identically distributed (i. i. d.) random variables obeying the lognormal probability density function (PDF):

$$P(W) = LN(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}W} \exp\left(-\frac{(\ln W - \mu)^2}{2\sigma^2}\right), \tag{8}$$

with σ^2, μ being the variance and the mean of $\ln(W)$, correspondingly.

Secondly, the three sets of wavelet coefficients $\{c^{\alpha}\}$ are calculated from coefficients $\{d\}$ obtained in the previous step as follows:

$$c^1_{j,m,n} = d_{j,m,n}\cos(\varphi)\cos(\theta)$$

$$c^2_{j,m,n} = d_{j,m,n}\sin(\varphi)\cos(\theta) \quad (9)$$

$$c^3_{j,m,n} = d_{j,m,n}\sin(\theta)\ ,$$

here $\varphi \in [-\pi, \pi]$, $\cos\theta \in [-1,1]$ are the uniform i. i. d. random numbers. This decomposition corresponds to the uniform distribution of the directions of the vector $(c^1_{j,m,n}, c^2_{j,m,n}, c^3_{j,m,n})$ of the length $d_{j,m,n}$.

After the wavelet coefficients have been generated, the phase is finally produced by the inverse wavelet transform (5). It is rigorously proved in [6] that the stochastic functions built this way obey the multifractal scaling

$$S_q(L) \sim \left(\frac{L}{L_0}\right)^{-\frac{\mu q}{ln2}-\frac{\sigma^2 q^2}{2ln2}}, \quad (10)$$

and this result is independent of the type of the wavelet used. Therefore, setting the parameters of the lognormal distribution (8) employed in the generating procedure (6) as $\mu = -h\, ln2$ and $\sigma^2 = \lambda^2\, ln2$, we are capable of producing the stochastic functions with scaling defined by (4). After appropriate normalization, the functions generated this way are then utilized as the intermittent non-Gaussian random phase distribution on the screen.

To demonstrate the non-Gaussian behavior of the stochastic distributions generated this way, in the Figure 1 the PDFs of the phase differences over the fixed spatial separation are plotted for different values of the intermittency parameter λ^2.

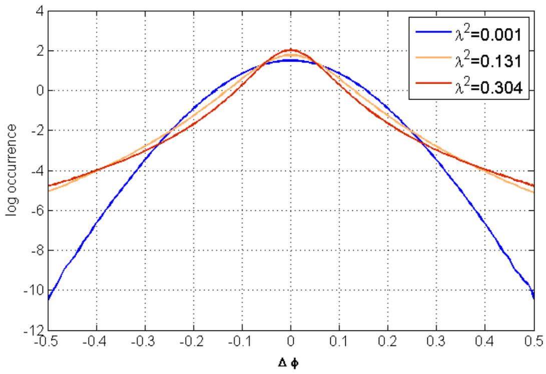

Figure 1. The set of logarithms of PDFs of phase differences corresponding to some different values of the intermittency parameter λ^2.

As is seen from the Figure 1, the logarithm of PDF for the small value of λ^2 has a parabolic shape which fits the normal distribution; the curves for bigger λ^2 remarkably depart from parabola thus demonstrating non-Gaussian shapes.

3. Simulations and numerical results

After the stochastic phase screen is specified, the complex amplitude of the field of the initially plane wave after passing the screen is given by the Kirchhoff integral which is used in the Fresnel approximation

$$u(\boldsymbol{r}, z) = -\frac{ik}{2\pi Z}\iint \exp\left(i\left[\varphi(\boldsymbol{r}') + \frac{k}{2Z}|\boldsymbol{r}-\boldsymbol{r}'|^2\right]\right) d^2\boldsymbol{r}'. \quad (11)$$

Here k is the wavenumber, Z is the distance from the screen. The double integration in (11) is accomplished numerically employing the fast Fourier transform algorithm, producing the values of the complex amplitude of the field on the plane behind the phase screen.

Since the paper aims to assess the effects of intermittent fluctuations on the scattered field, in the following we are going to compare several statistical characteristics of the scattered field for different extents of departure of the stochastical phase distribution on the screen from the Gaussianity. This departure will be characterized by the parameter λ^2 of the scaling exponents (4). For the case of the Gaussian random fields, the structure function of the second order provides the full statistical description of the fluctuations, being proportional to the spatial separation to the power of $2H$. For the scaling exponents (4) this corresponds to the case of $h = H, \lambda^2 = 0$. If in the general non-Gaussian case $(\lambda^2 > 0)$ the value of $h - \lambda^2 = H$ is being kept constant, then the second-order structure function or, equivalently, the power spectrum of the stochastic function will be the same for an arbitrary value of λ^2. This means that the stochastic distributions of phase possessing the same second order structure functions and spectra may well have different extent of intermittency.

In the following numerical experiments, the stochastic distributions of phase on the screen are generated for different values of the intermittency parameter λ^2, but having the identical spatial spectra and second order structure functions by keeping $h - \lambda^2 = H, H = \frac{5}{6}$. Other parameters of the simulations are as follows: the screen contains 1024x1024 points, the grid step is $10\,m$, the standard deviation of phase on the screen is 4.37 and is kept constant in the simulations. Frequency is 10^9 Hz.

Firstly, consider the coherence function of the complex field $u(\boldsymbol{r}, z)$ on the distance z from the random phase screen

$$\Gamma(L) = \langle u(\boldsymbol{r})u^*(\boldsymbol{r}+L)\rangle = \langle e^{i(\varphi(\boldsymbol{r})-\varphi(\boldsymbol{r}+L))}\rangle. \quad (12)$$

Equation (12) reflects the well-known fact that the coherence function of the plane wave behind the phase screen does not change with the distance. For the Gaussian phase screen it is expressed through the second order structure function of phase on the screen

$$\Gamma(L) = e^{-\frac{1}{2}S_2(L)}. \quad (13)$$

In a general case of non-Gaussian phase distribution, the averaging in (12) is performed numerically for different values of λ^2. The results of simulations are plotted in the Figure 2, each point is obtained by averaging over 1000 realizations of the phase screen.

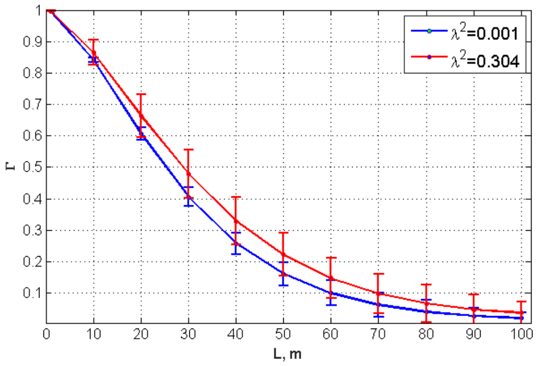

Figure 2. The set of coherence functions corresponding to the different values of the intermittency parameter $\boldsymbol{\lambda^2}$.

It is clearly seen in the Figure 2 that the coherence curves corresponding to the larger intermittency parameters lie higher than those for the smaller λ^2. In other words, it appears that the coherence of the field increases as the intermittency rises. Note, that the power spectrum of the stochastic phase on the screen is the same for all three curves.

Let now consider the amplitude scintillation behind the screen. It is commonly accepted to characterize the severity of the amplitude or intensity scintillation by the scintillation index S_4 that is the normalized variance of the intensity scintillation

$$S_4 = \sqrt{\frac{\langle|u|^4\rangle - \langle|u|^2\rangle^2}{\langle|u|^2\rangle^2}}. \qquad (14)$$

In the Figure 3, the dependencies of the scintillation index on the distance from the screen are plotted for the different values of the intermittency parameter.

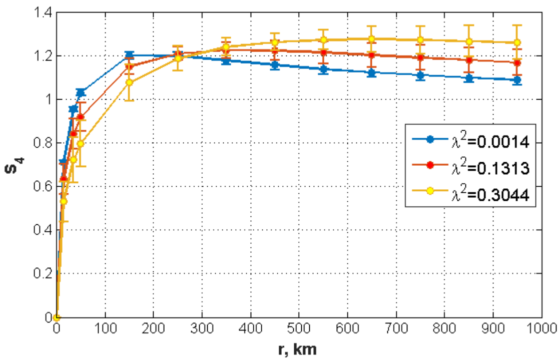

Figure 3. The dependencies of the scintillation index S_4 on the distance r from the screen for the different values of the intermittency parameter λ^2. Here $\sigma_\varphi = 12.35$, frequency is $5 \cdot 10^8$ Hz.

Qualitatively, all three curves behave similarly: they rise, reach their maxima and then go down, slowly approaching unity. But the higher is the intermittency parameter λ^2, the more slow is the dependence of S_4 on the distance. Once again, the power spectrum and the structure function of the second order of the stochastic phase on the screen are the same for all the three curves.

To conclude, this paper presents the technique of accounting for the intermittency of the ionospheric electron density irregularities on the statistical characteristics of the fluctuations of the transionospheric field. The initial numerical results show deviations of the behavior of the statistical moments of the field from those in non-intermittent Gaussian regime.

5. References

1. A. W. Wernik, M. Grzesiak, "Scintillation caused by the ionosphere with non-Gaussian statistics of irregularities," *Radio Science,* **46**, 6, 2011, pp. 1-9, doi: 10.1029/2011RS004716.

2. K. C. Yeh, C.-H. Liu, "Radio wave scintillations in the ionosphere," *Proceedings of the IEEE*, **70***,* 4, 1982, pp.324–360, doi:10.1109/PROC.1982.12313.

3. K.R. Sreenivasan, "Fractals and Multifractals in Fluid Turbulence," *Annual Review of Fluid Mechanics*, **23**, January 1991, pp. 539-604, doi: 10.1146/annurev.fl.23.010191.002543.

4. S. Boldyrev, C. R. Gwinn, "Radio-wave propagation in the non-Gaussian interstellar medium," *The Astrophysical Journal*, 624, May 2005, pp. 213–222, doi: 10.1086/428919.

5. E. Jakeman, K.D. Ridley, "Modeling fluctuations in scattered waves", *Taylor and Francis*, 2006.

6. A. Arneodo, E. Bacry, J. F. Muzy, "Random cascades on wavelet dyadic trees," *Journal of Mathematical Physics*, **39**, 8, 1998, pp. 4142-4164, doi: 10.1063/1.532489.

7. N. Decoster, S. Roux, A. Arnéodo, "A wavelet-based method for multifractal image analysis. II. Applications to synthetic multifractal rough surfaces," *The European Physical Journal B - Condensed Matter and Complex Systems,* **15**, 4, June 2000, pp. 739-764, doi: 10.1007/s100510051179.