Integral representation for two-frequency coherence function of ionospheric radio signal

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Abstract

The double-weighted Fourier transform method, proposed for describing wave fields in a multiscale inhomogeneous medium, is adopted to derive a two-frequency two-position coherence function. It is shown that, as in the Markov approximation, the spatial coherence function coincides with the geometrical optics approximation. The resulting condition for validity of the geometrical optics approximation for frequency coherence is weaker than the same condition for individual realizations, especially for narrowband signals.

1. Introduction

The development of communication systems and remote sensing of Earth and space environment faces the problem of distortion of a radio signal propagating through such randomly inhomogeneous media as the ionosphere and lower atmosphere. Much research is devoted to methods of calculating the space-time signal structure under such conditions. In view of the multiscale nature of the inhomogeneous signal propagation medium, different calculation methods are used [1]. The geometrical optics (GO) description of a field in a smoothly inhomogeneous medium should be assigned to the simplest and most popular representations [1]. However, it does not quite correctly describe the presence of irregularities with sizes smaller than the Fresnel radius. Diffraction effects caused by such irregularities were taken into account using methods of smooth perturbations [1-2], phase screens [3-4], and path integrals [5]. The most widely used method is the parabolic equation method (Markov approximation) [1], with which, assuming delta-correlation of the refractive index along a propagation path, we derive equations for field moments in a randomly inhomogeneous medium. In this approximation, equations for the coherence function are solved numerically and through application of methods for modal decomposition [6], two-scale expansions [7], and other asymptotic methods [8].

In [9-10], an expression was proposed for the wave field in a smoothly inhomogeneous medium in the form of the Double-Weighted Fourier Transform (DWFT). This integral representation is transformed into results of the GO approximation, methods of smooth perturbations and phase screens in domains of their applicability. Here, we adopt the DWFT method for constructing frequency coherence functions.

2. The DWFT method in statistical problems

To analyze the operation of communication, radar, navigation, and other radio systems using radio wave propagation in randomly inhomogeneous media, we should know the two-point two-frequency coherence function, which for spherical waves in the DWFT approximation takes the form

$$\begin{aligned}\Gamma_E\left(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega_1,\omega_2\right)&=\left\langle E\left(\boldsymbol{\rho}_1,\omega_1,\mathrm{z},\mathrm{z}_0\right)E^*\left(\boldsymbol{\rho}_2,\omega_2,\mathrm{z},\mathrm{z}_0\right)\right\rangle\\&=E_0\left(\boldsymbol{\rho}_1,\omega_1\right)E_0{}^*\left(\boldsymbol{\rho}_2,\omega_2\right)\widehat{\Gamma}\left(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega_1,\omega_2\right),\end{aligned}\tag{1}$$

where

$$\begin{aligned}&\widehat{\Gamma}\left(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega_1,\omega_2\right)=\widehat{\Gamma}\left(\Delta\boldsymbol{\rho},\Delta\boldsymbol{\rho}_0,\mathrm{k}_0,\mathrm{X}\right)\\&=\left[\widehat{k}_0\left(z-z_0\right)/(2\pi)\right]^2\exp\left\{i\widehat{k}_0\Delta\boldsymbol{\rho}\Delta\boldsymbol{\rho}_0/\left(z-z_0\right)\right\}\\&\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^2s\,d^2p\exp\left\{-i\widehat{k}_{\mathbf{0}}\left[\mathbf{ps}\left(z-\mathrm{z}_0\right)+\mathbf{s}\Delta\boldsymbol{\rho}_0-\mathbf{p}\Delta\boldsymbol{\rho}\right]\right.\\&\left.-D(\mathbf{p},\mathbf{s},X)/2\right\},\end{aligned}\tag{2}$$

$$\begin{aligned}D(\mathbf{p},\mathbf{s},X)=\frac{0.5k_p^4}{\left[k_0\left(1-X^2\right)\right]^2}\int_{z_0}^{z}\frac{\sigma_N^2(\eta)}{\bar{N}^2}\Big[\left(1+X^2\right)\\\times\bar{\Psi}_N\left(0,\eta\right)-\left(1-X^2\right)\bar{\Psi}_N\left(\bar{\boldsymbol{\rho}}(\mathbf{p},\mathbf{s},\eta),\eta\right)\Big]d\eta,\end{aligned}\tag{3}$$

$k_i=\omega_i/c=2\pi/\lambda_i$, c , λ_i , and $E_0(\boldsymbol{\rho},z)$ are the wave number, speed of light, wavelength, and wave field in free space respectively; $k_p=\omega_p/c=2\pi\sqrt{80.6\bar{N}}/c$ is the wave number corresponding to the frequency ω_p of "background" plasma with the electron density $\bar{N}$;
$\sigma_N^2(\eta)\bar{\Psi}_N\left(\boldsymbol{\rho}(\eta),\eta\right)=\int_{-\infty}^{\infty}\Psi_N\left(\boldsymbol{\rho}(\xi,\eta),\xi,\boldsymbol{\rho}_\eta(\xi,\eta),\eta\right)d\xi$;
$\Delta\boldsymbol{\rho}=\boldsymbol{\rho}_1-\boldsymbol{\rho}_2$, $\Delta\boldsymbol{\rho}_0=\boldsymbol{\rho}_{01}-\boldsymbol{\rho}_{02}$, $\boldsymbol{\rho}_\eta=\left(\boldsymbol{\rho}_1+\boldsymbol{\rho}_2\right)/2,\xi=z_1\text{-}\mathrm{z}_2$,
$k_0=\left(k_1+k_2\right)/2$, $X=\left(k_2-k_1\right)/2k_0$, $\widehat{k}_0=k_0\left(X^{-1}-X\right)$;
σ_N^2 and $\Psi_N\left(\Delta\boldsymbol{\rho},\xi,\boldsymbol{\rho}_\eta,\eta\right)=\left\langle\tilde{N}\left(\boldsymbol{\rho}_1,\mathrm{z}_1\right)\tilde{N}\left(\boldsymbol{\rho}_2,\mathrm{z}_2\right)\right\rangle$ are the variance and correlation function of statistically quasi-

homogeneous fluctuations of electron density of ionospheric plasma.

To derive a spatial coherence function from (2), we assume that frequency separation $X=0$ is zero. This allows us to calculate integrals in (2) and get

$$\widehat{\Gamma}\left(\Delta\boldsymbol{\rho},\Delta\boldsymbol{\rho}_0,\mathrm{k}_0,0\right)=\exp\left\{-D(\Delta\boldsymbol{\rho},\Delta\boldsymbol{\rho}_0,0)/2\right\}\ , \qquad (4)$$

where for $D(\Delta\boldsymbol{\rho},\Delta\boldsymbol{\rho}_0,0)$ we should use (3) with $X=0$, and

$$\begin{aligned}&\overline{\boldsymbol{\rho}}(\mathbf{p},\mathbf{s},\eta)=\overline{\boldsymbol{\rho}}\left(\Delta\boldsymbol{\rho},\Delta\boldsymbol{\rho}_0,\eta\right)\\&=\left[\Delta\boldsymbol{\rho}_0(\mathrm{z}-\eta)+\Delta\boldsymbol{\rho}(\eta-\mathrm{z}_0)\right]/\left(\mathrm{z}-\mathrm{z}_0\right).\end{aligned} \qquad (5)$$

Solution (4)–(5) can be found using the Markov and GO approximations [1].

Now, by putting $\Delta\boldsymbol{\rho}=\Delta\boldsymbol{\rho}_0=0$ in (2), we obtain the frequency coherence function

$$\begin{aligned}&\widehat{\Gamma}_{\omega}\left(\mathrm{k}_0,\mathrm{X}\right)=\widehat{\Gamma}\left(0,0,\mathrm{k}_0,\mathrm{X}\right)=\left[\widehat{k}_0\left(z-z_0\right)/(2\pi)\right]^2\\&\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^2s\,d^2p\exp\left\{-i\widehat{k}_0\mathbf{ps}\left(z-\mathrm{z}_0\right)-D(\mathbf{p},\mathbf{s},X)/2\right\}.\end{aligned} \qquad (6)$$

If the condition

$$\widehat{r}_{fr}<l_N\ , \qquad (7)$$

holds, where l_N is the characteristic scale of irregularities,

$$\widehat{r}_{fr}=\left(z-z'\right)\left(z'-z_0\right)/\left[\widehat{k}_0\left(z-z_0\right)\right]=\frac{r_{fr}}{X^{-1}-X}<r_{fr} \qquad (8)$$

and r_{fr} is the Fresnel radius, the asymptotic calculation of integrals in (6) yields

$$\widehat{\Gamma}_{\omega}\left(\mathrm{k}_0,\mathrm{X}\right)\approx\exp\left\{-\frac{\mathrm{X}^2\int_{z_0}^{z}k_p^4\sigma_N^2(\eta)/\overline{N}^2\overline{\Psi}_N(0,\eta)\,d\eta}{2\left[k_0\left(1-X^2\right)\right]^2}\right\}. \qquad (9)$$

It is interesting that condition (7) for GO approximation (9) of the frequency coherence function is weaker than the condition for validity of this approximation without field averaging.

3. Simulation of the frequency coherence function

First, we use a model of random irregularity with a Gaussian spectrum:

$$\Phi_N\left(\boldsymbol{\kappa}\right)=L_0^3\sigma_N^2\pi^{-3/2}\exp\left[-L_0^2\kappa^2\right] \qquad (10)$$

For such a spectrum

$$\overline{\Psi}_N\left(\boldsymbol{\rho},\eta\right)=\sigma_N^2(\eta)2\sqrt{\pi}L_0\exp\left[-\rho^2/\left(4L_0^2\right)\right] \qquad (11)$$

Substituting (11) in (3), we get

$$\begin{aligned}&D(\mathbf{p},\mathbf{s},X)=\frac{\sqrt{\pi}L_0}{\left[k_0\left(1-X^2\right)\right]^2}\int_{z_0}^{z}k_p^4\frac{\sigma_N^2(\eta)}{\overline{N}^2}\Big[\big(1\\&+X^2\big)-\left(1-X^2\right)\exp\left[-\left|\overline{\boldsymbol{\rho}}(\mathbf{p},\mathbf{s},\eta)\right|^2/\left(4L_0^2\right)\right]\Big]d\eta.\end{aligned} \qquad (12)$$

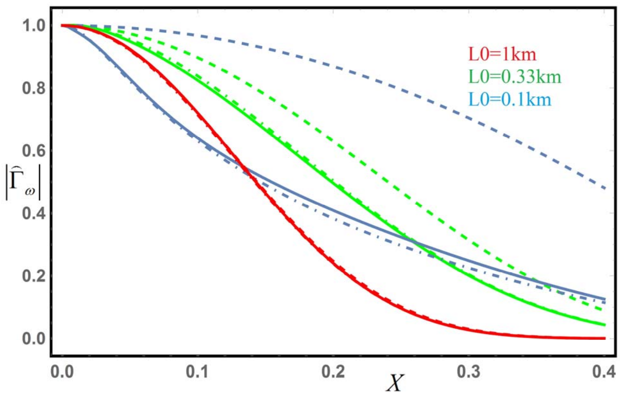

Figure 1. Modulus of frequency coherence function for the Gaussian spectrum of irregularities with different scales

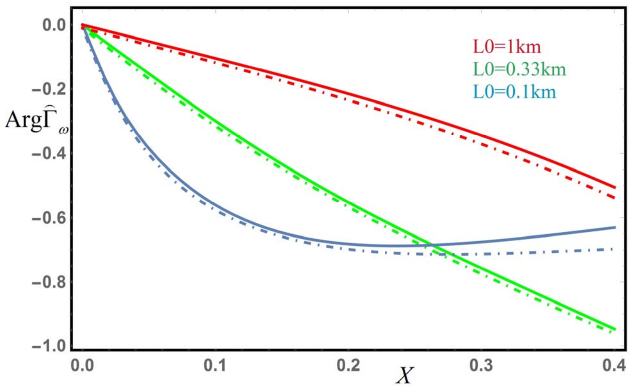

Figure 2. Argument of frequency coherence function for the Gaussian spectrum of irregularities with different scales

The results of simulation from (6), (12) with an observer on the plane $z=z_0=0$ and with a source on the plane $z=800\,\mathrm{km}$ for the scales $L_0=100\ \mathrm{m}$, $330\ \mathrm{m}$, and $1\ \mathrm{km}$ are presented in Figures 1, 2. A statistically homogeneous layer with random irregularities was located between 150 and 350 km. Fluctuations were taken to be 5 % of the background media ($\sigma_N/\overline{N}=0.05$) whose plasma frequency was assumed to be constant and equal to 6 MHz. In the Figures, in addition to calculations with formulas (6) and (12), the dashed line shows calculations with GO formula (9). The Figures demonstrate that in the GO approximation the channel bandwidth is wider, i.e. diffraction effects narrow the bandwidth.

The modified Karman spectrum [1-2] and Shkarofsky's model [11] are more appropriate to the real turbulence spectrum. Here we present simulation results for the latter model, in which

$$\bar{\Psi}_N(\mathbf{r}) = \frac{\left(\kappa_0\sqrt{\rho^2+l_m^2}\right)^{(p-2)/2} K_{(p-2)/2}\left(\kappa_0\sqrt{r^2+l_m^2}\right)}{\kappa_0\left(\kappa_0 l_m\right)^{(p-3)/2} K_{(p-3)/2}\left(\kappa_0 l_m\right)}, \quad (13)$$

where $\kappa_0 = 2\pi / L_0$, L_0 , and l_m are outer and inner scales respectively, $K_\mu(z)$ is a modified Bessel function (Hankel function of imaginary argument).

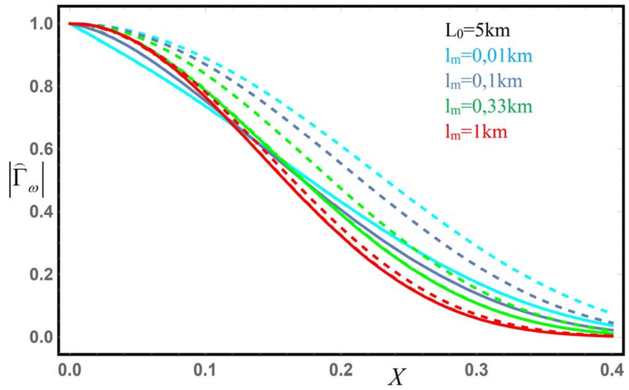

Figure 3. Modulus of the frequency coherence function for Shkarofsky spectrum with different inner scales.

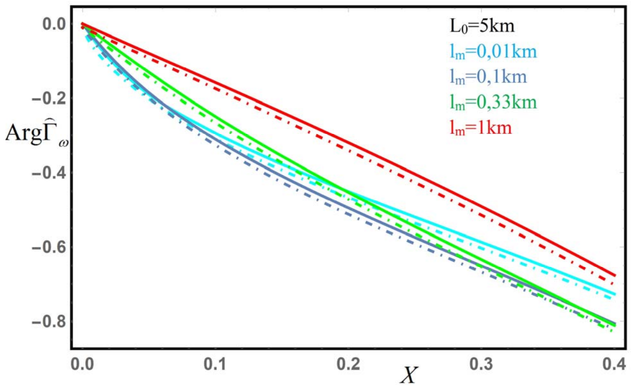

Figure 4. Argument of frequency coherence function for Shkarofsky spectrum with different inner scales.

Formulas (6) and (3) in view of (13) have been used for the numerical simulation. Simulation results for $p = 11/3$, the outer scale $L_0 = 5\,\text{km}$ and for various inner scales $l_m = 0.01\,\text{km},\ 0.1\,\text{km},\ 0.33\,\text{km}$, and $1\,\text{km}$ are shown in Figures 3 and 4. The value $l_m = 0.01\,\text{km}$, which is lower than usual inner scales, corresponds to the Karman spectrum into which the Shkarofsky spectrum is transformed at $l_m \to 0$ [11].

Figures 5 and 6 present the frequency coherence versus the outer scale L_0 . For this purpose, with the inner scale $l_m = 0.1\,\text{km}$ we take various outer scales $L_0 = 1\,\text{km},\ 5\,\text{km},\ 10\,\text{km}$.

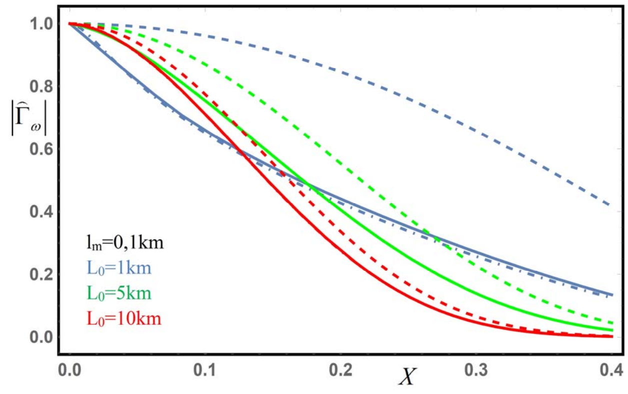

Figure 5. Modulus of the frequency coherence function for the Shkarofsky spectrum with different outer scales.

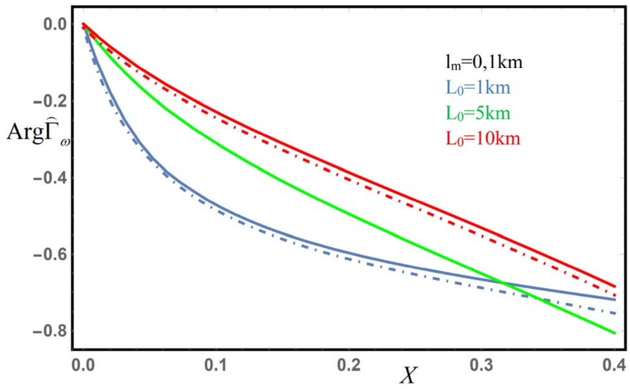

Figure 6. Argument of the frequency coherence function for the Shkarofsky spectrum with different outer scales.

4. Frequency coherence of waves after passing through remote irregularities

The complexity of the computations from the derived formulas is related to multiple integrals. In [13], at a large distance between irregularities and a source/observer in DWFT solution (2), the variables were changed, thus allowing asymptotic calculation of the integral over one of the surfaces. After changing to $\boldsymbol{\rho}_b\ \mathbf{p}_s$ in (2)–(3)

$$\begin{cases} \mathbf{s} = \left(\boldsymbol{\rho}_b - (z - z_b)\mathbf{p}_s\right)/(z - z_0) \\ \mathbf{p} = -\left(\boldsymbol{\rho}_b + (z_b - z_0)\mathbf{p}_s\right)/(z - z_0), \end{cases} \quad (14)$$

we get

$$\begin{aligned} \hat{\Gamma}_\omega(\mathrm{k}_0, \mathrm{X}) &= \left[\hat{k}_0/(2\pi)\right]^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2p_s d^2\rho_b \\ &\times \exp\left\{ -i\hat{k}_0\left(\mathbf{p}_s - \mathbf{p}_{s0}\right)^2 \frac{(z - z_b)(z_b - z_0)}{(z - z_0)} \right. \\ &\left. + i\frac{1}{4}\hat{k}_0 \boldsymbol{\rho}_b{}^2 \frac{(z - z_0)}{(z - z_b)(z_b - z_0)} - \frac{1}{2}D(\mathbf{p}_s, \boldsymbol{\rho}_b) \right\}, \end{aligned} \quad (15)$$

where $\mathbf{p}_{s0} = \boldsymbol{\rho}_b\left[(z - z_b)^{-1} - (z_b - z_0)^{-1}\right]$ and $D(\mathbf{p}_s, \boldsymbol{\rho}_b)$ are determined by (3), in which we should insert

$$\bar{\boldsymbol{\rho}}(\boldsymbol{\rho}_b,\mathbf{p}_s,\eta)=\boldsymbol{\rho}_b+\mathbf{p}_s(z_b-\eta)\ . \qquad (16)$$

For irregularities located far from a source and an observer, the dependence of $D(\mathbf{p}_s,\boldsymbol{\rho}_b)$ on $\mathbf{p}_s$ is weak, the integral over $\mathbf{p}_s$ in (15) is calculated asymptotically, and we have

$$\begin{aligned}\hat{\Gamma}_\omega(\mathrm{k}_0,\mathrm{X}) &= \frac{-i\hat{k}_0(z-\mathrm{z}_0)}{4\pi(z-\mathrm{z}_b)(z_b-\mathrm{z}_0)}\\ &\times\int_{-\infty}^{\infty} d^2\rho_b \exp\left\{i\frac{1}{4}\frac{\hat{k}_0\boldsymbol{\rho}_b{}^2(z-\mathrm{z}_0)}{(z-\mathrm{z}_b)(z_b-\mathrm{z}_0)}-\frac{1}{2}D(\boldsymbol{\rho}_b)\right\},\end{aligned} \qquad (17)$$

where $D(\boldsymbol{\rho}_b)$ is determined by (3) after substituting

$$\bar{\boldsymbol{\rho}}(\boldsymbol{\rho}_b,\eta)=\frac{\boldsymbol{\rho}_b}{2}\left[(z-\eta)/(z-z_b)+(\eta-z_0)/(z_b-z_0)\right] \quad (18)$$

In the above Figures, dash-and-dot lines show the results of calculations with approximate formula (17). Of note is the high accuracy of the approximate formulas even if irregularities are not so far.

5. Conclusion

Using the DWFT method, we have derived a formula for two-frequency two-position coherence function for two spherical waves. We have obtained condition (7) for transformation of this formula to the GO approximation, which in narrowband systems is much weaker than the corresponding condition for individual realizations. We have presented the results of calculations of the frequency coherence function when random irregularities are described by the Gaussian spectrum and Shkarofsky's model. For remote irregularities, the formula for double weighted transform is reduced to a single weighted Fourier transform (Fresnel transform) through a change of variables and asymptotic integration.

6. Acknowledgments

This work was supported by the Government of the Russian Federation for Assignment for Scientific Research from the Ministry of Education and Science, Russia (No. 3.903.2017 / 4.6). I am very grateful to O.A. Kulish for her assistance in preparing the English version of the manuscript.

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