Analytical Approximations of Low-Frequency Scattering from Homogeneous Spheres

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Because of its perfect isotropy, an electrically small homogeneous sphere is an important building block in the design and construction of artificial electromagnetic materials, including metamaterials, over a broadest range of frequencies, from microwave and terahertz to optical. The relative permittivity ε_r of the sphere may, therefore, broadly vary as its real part can be positive (all natural materials at microwave frequencies) or negative (noble metals at optical frequencies), the absolute value large or small and the imaginary part can correspond to a strongly absorbing ($|\mathrm{Im}\,\varepsilon_r| > 1$) or an almost lossless ($\mathrm{Im}\,\varepsilon_r \to 0$) material. The relative permeability μ_r may also deviate from unity if the material is magnetic. Though exact and low-frequency analytical solutions for scattering from homogeneous spheres have been available since more than a century, e.g. (J.A. Stratton, *Electromagnetic Theory*, 1941, pp. 563-573), a complete description of the low-frequency scattering for the whole variety of possible material parameters is still missing. This work closes the gap by providing a complete set of analytical low-frequency approximations on the complex plane of ε_r.

We start with the exact (multipole) series solutions for a homogeneous sphere of the radius a, including the solutions for the total scattering, extinction and absorption cross sections, σ_{T}, σ_{ext} and σ_{abs}. In the case of an electrically small sphere ($ka \ll 1$), k being the wavenumber in free space, the electric dipole term typically dominates, and approximating it up to the third power of ka, which is the leading order, results in the well-known quasi-static approximation. The quasi-static approximation is, however, not always accurate. For example, if the losses in the material of the sphere are small, then approximating up to the sixth power of ka in the electric dipole term is necessary. Furthermore, the magnetic dipole term may become important for magnetic spheres or for non-magnetic spheres with large values of $|\varepsilon_r|$. Further special cases arise when ε_r is close to a pole (resonance) of a multipole expansion coefficient. The poles are located in the upper (unphysical) half-plane on the complex ε_r plane but approach the physical, lower half-plane for electrically small spheres. In a vicinity of these poles, special approximations of the multipole contributions to the scattered field are necessary. We show that a tear of the poles is located close to the interval $-2 < \varepsilon_r < -1$, accumulating at $\varepsilon_r = -1$, while another tear follows the semi-axis $(\pi / ka)^2 < \varepsilon_r < +\infty$. We also provide the resonance approximations for the mulitpole expansion coefficients and show that at the values of ε_r that are complex conjugates of the resonant values, the absorption cross section has a sharp maximum and satisfies the relations:

$$\sigma_{\mathrm{abs}} \approx \sigma_{\mathrm{T}} \approx \frac{1}{2}\sigma_{\mathrm{ext}} \approx \frac{\pi}{2k^2}(2n+1)$$

where n is the multipole order ($n = 1, 2, \ldots$). The derived analytical approximations may significantly help in tuning artificial materials with spherical inclusions to a desired scattering behavior.