URSI-B EMTS 2025, Bologna, 23–27 June

Electromagnetic wave-based solutions to partial differential equations using networks of waveguide-based metatronic T-circuits

Ross Glyn MacDonald(1), Alex Yakovlev(2), Christian Johnson-Richards(1) and Victor Pacheco-Peña(1)
(1) School of Mathematics, Statistics and Physics, Newcastle University, Newcastle Upon Tyne, NE1 7RU, United Kingdom;
(2) School of Engineering, Newcastle University, Newcastle Upon Tyne, NE1 7RU, United Kingdom

Abstract

Here, an electromagnetic (EM) wave-based method of calculating the solutions to partial differential equations is presented. This is done by exploiting a network of waveguide-based metatronic T-circuits arranged in a grid. It will be shown how this technique can enable discretized approximate solutions to partial differential equations in the form of the Helmholtz equation. This includes the possibility of producing solutions to both Dirichlet boundary value and open boundary problems.

1. Introduction

Partial differential equations (PDEs) are fundamental to many field (from mathematics to physics to engineering). Indeed, they can be used to model a wide range of phenomena such as the propagation of waves[1] and thermal radiation[2], as examples. To solve PDEs, one can make use of software-based approaches such as the finite difference[3] or finite element method[4] (FEM) to produce approximate numerical results. However, would it be possible to do this hardware-based using EM waves? If this is possible, calculations could be done at the speed of light within the materials. This may enable faster calculations avoiding highly computational techniques [5, 6].

Analogue computing with EM waves has emerged as a promising computing paradigm[7]. This approach aims to harness the potential of waves for high-speed[8] computing with the possibility of parallelisation[9]. EM wave-based analogue processors have been proposed and demonstrated to perform mathematical operations such as differentiation[10-12], integration[8, 13], and convolution [8, 14] as well as matrix inversion[15] and the calculation of solutions to integral equations[16, 17] or PDEs[18-20], to name a few[21-25].

Early analogue computers exploited grid-like networks of lumped circuit elements to produce approximate PDE solutions[26, 27]. This was done by drawing an analogy between the voltage and current distributions of the network (sampled at the junctions between lumped elements) and the finite difference representations of the PDEs to be solved[26, 27]. Recently, metatronic circuits[28] have been introduced as a powerful technology capable of emulating the performance of electrical circuits using EM waves at high frequencies[29, 30]. This has shown to be pivotal in allowing known circuit theory to be efficiently translated to new domains. For instance, to aid in the design of structures such as filters[29, 31], antennas[32] and sensors[33, 34], to name a few. Indeed, metatronic circuits have also been applied alongside epsilon-near-zero (ENZ) media to produce approximate solutions to PDEs in the form of the Poisson equation[19].

Motivated by the potential applications of metatronic circuits in the realm of EM wave-based analogue computing, in this study we present a structure capable of solving PDEs in the form of the Helmholtz equation. This is done by using a network of waveguide-based metatronic circuits connected at junctions in a series configuration. In so doing, ENZ host media emulating interconnects are not required. As will be shown, the parameters of the PDE being solved may be controlled by tailoring the effective impedance values of the various metatronic circuits. This technique is applied to solve both Dirichlet and open boundary value problems.

2. Results and discussion

To understand how metatronic circuits may be exploited for PDE solving, first consider the conventional circuit presented in Fig. 1a. This circuit is a grid-like network of T-circuits connected in series at a junction. Due to this, any flow of current through the T-circuits will produce a rotating current I_a around the center of each junction. Here $I_a = 0, 1, 2, 3, 4$ is the current flowing around the center, top, right, bottom, and left junctions in Fig. 1a, respectively. Then by considering the division of current though the series and parallel impedances of the connecting T-circuits (Z_s and Z_p, respectively) and applying Kirchhoff's voltage law at the central junction the following equation is revealed[20]:

$$Z_p(I_1 + I_2 + I_3 + I_4 - 4I_0) - 4Z_s I_0 = 0 \qquad (1)$$

which describes the relationship between I_0 and the currents around the connected junctions I_1, I_2, I_3 and I_4. Importantly, if the impedance values of each connecting T-circuit (Z_s and Z_p) are all chosen to be the same, then Eq. 1 will hold for any junction in the network which is connected to it 4 nearest neighboring junctions. This is true for all junction in the network, except for the outermost *boundary junctions* at the edges of the network which are connected to at most 3 other junctions (as can be seen in Fig. 1b). Considering this and comparing Eq. 1 to the well-known formula for the finite difference representation of the Helmholtz equation in 2D[3] $\{[g(x+h,y)+g(x-h,y)+g(x,y-h)+g(x,y+h)-4g(x,y)]/h^2+k^2g(x,y)=0$ where x, y are the independent variables of the function g, k is the wavenumber and h is the finite difference step size as shown in Fig. 1b$\}$, it can be seen that the two expressions are analogous with $h=1/\sqrt{Z_p}$ and $k=2\sqrt{-Z_s}$. Using this analogy, the distribution of current flowing around each junction resembles a discretized solution to the Helmholtz equation. It should be noted that using this analogy can produce a complex h value, which is undesirable for most applications, however this can be avoided by transforming the observed currents with $I'_a = I_a/Z_p^*$ and selecting, $h=1/|Z_p|$ and $k=2\sqrt{-Z_sZ_p^*}$, to achieve the same analogy.

Now, how can one implement the structure presented in Fig. 1a? Here we answer this question by exploiting metatronic circuits being emulated by the waveguide-based structure shown in Fig. 1c. Three thin dielectric (or metallic) slabs are inserted into a metallic parallel plate waveguide. These slabs are separated by a distance of $\lambda_h/4$ (λ_h is the wavelength within the host medium, i.e. the wavelength within the parallel plate waveguide). The first (left) and last (right) slabs are also separated from the ends of the waveguide by a distance $\lambda_h/4$. With this setup, the entire waveguide structure emulates the behavior of a T-circuit with the widths and permittivities of the outer and inner slabs (blue and red slabs in Fig. 1c) determining the emulated Z_s and Z_p impedance values[20, 29, 35]. These metatronic T-circuits are then arranged into a grid and connected in series to emulate the network shown in Fig. 1a. The calculated PDE solution can then be extracted from the out-of-plane magnetic field at the center of each waveguide junction.

To demonstrate the potential of this PDE solving technique, an example of a 25×25 network of waveguide-based metatronic T-circuits solving a Dirichlet boundary value problem is shown in Fig. 1d. In this example the width and permittivities of the dielectric slabs have been chosen so that the emulated impedance values are $Z_p=5iZ_0$ and $Z_s=-0.45iZ_0$, where Z_0 is the characteristic impedance of the parallel plate waveguides. These values correspond to the following PDE parameters: $h=0.2$ and $k=3$. As will be shown in detail during the conference, the Dirichlet boundary conditions are then implemented via external signals applied at the *boundary junctions*. In this example the boundary conditions are chosen such that the magnitude of the field at each *boundary junction* is the same while the phase varies by 2π anti-clockwise around the edge of the network. Numerical simulation results (calculated using the commercial simulation software CST Studio Suite®) for the out-of-plane magnetic field distribution of the PDE solving network are presented in Fig. 1d along with a theoretical solution to the Dirichlet boundary problem using an FEM MATLAB® tool to solve PDEs[36]. As it can be seen, there is a clear agreement between the two solutions indicating the PDE solving network is indeed producing a valid solution to the boundary value problem.

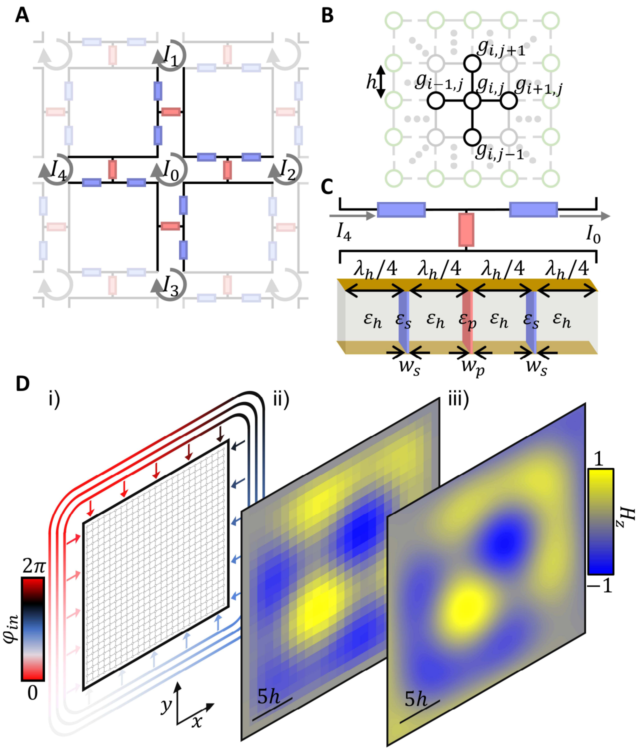

Figure 1. a. Transmission line representation of the PDE solving network using series (blue) and parallel (red) impedances. **b**. Finite difference grid representation of the structure presented in **a** with *boudary juncitons* (green) . **c**. Schematic representation of a metatronic T-circuit. **d**. Solution of a closed boundary value problem using a $\mathbf{25\times25}$ network. **i.** schematic representation of the PDE solving network and the implemented boundary conditions. **ii.** , **iii.** calculated PDE solution extracted from the PDE solving network and an FEM simulation tool (MATLAB®), respectivley.

Using this technique, it is also possible to solve for open boundary problems such as the focusing or scattering of waves due to a lens or obstacle[20]. As will be shown in the conference, this is done by extending the PDE solving network and allowing the incident signals to decay within. This is similar to the functionality of a perfectly matched layer (PML) in software-based PDE solvers. For completeness, an example is provided in Fig. 2b where the solution of a full 50×50 metatronic network using transmission line theory is presented (right) along with the solution of using the FEM tool (left). This example the scattering of a wave though a hole in a metal screen. The

PDE parameters of this network are the same as presented in Fig. 1d except for the regions marked in white where a metallic insert has been modeled by removing the connecting waveguide-based metatronic T-circuits in these regions. The modeled hole in the center of this reflective region has 7×10 junctions along the horizontal and vertical directions, respectively, which is approximately $2\lambda/3$ and λ, respectively (where λ is the wavelength of the PDE to be solved). A signal is then introduced to this structure by applying a constant boundary condition to the leftmost boundary junctions such that the input signal resembles a plane wave in the PDE solution. As observed, the results are in agreement, demonstrating the potential of the proposed structure to solve PDEs.

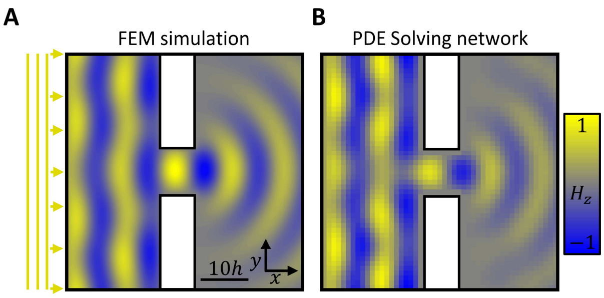

Figure 2. Example the solution of an open boundary value problem using a $\mathbf{50 \times 50}$ network. **a**, **b**. PDE solution from an FEM simulation tool (MATLAB®) and from the PDE solving network, respectivly.

3. Conclusions

To summarize, a method for PDE solving using EM waves has been presented by exploiting networks of waveguide-based metatronic T-circuits. Different examples have been shown including both Dirichlet boundary value and open boundary problems. These results may open new opportunities in EM wave-based analogue computing and the technique may be translated to different spectral regimes (from microwaves up to optical frequencies).

4. Acknowledgements

This work was supported by the Leverhulme Trust under the Leverhulme Trust Research Project Grant scheme (No. RPG-2020-316 and RPG-2023-024). For the purpose of Open Access, the authors have applied a CC BY public copyright license to any Author Accepted Manuscript (AAM) version arising from this submission.

References

[1] D. M. Pozar, *Microwave engineering*, 4th ed. Hoboken, NJ, USA: John wiley & sons, 2011.

[2] K. Nandakumaran, A and S. Datti, P, “Heat Equation,” in *Partial Differential Equations*, Cambridge University Press, 2020, pp. 216–251. doi: 10.1017/9781108839808.009.

[3] B. Fornberg, “Generation of Finite Difference Formulas on Arbitrarily Spaced Grids,” *Math Comput*, vol. 51, no. 184, p. 699, Oct. 1988, doi: 10.2307/2008770.

[4] Z. Bofang, *The Finite Element Method*, no. November 2020. Wiley, 2018. doi: 10.1002/9781119107323.

[5] S. Lew, C. H. Wolters, T. Dierkes, C. Röer, and R. S. MacLeod, “Accuracy and run-time comparison for different potential approaches and iterative solvers in finite element method based EEG source analysis,” *Applied Numerical Mathematics*, vol. 59, no. 8, pp. 1970–1988, Aug. 2009, doi: 10.1016/j.apnum.2009.02.006.

[6] M. F. Pellissetti and R. G. Ghanem, “Iterative solution of systems of linear equations arising in the context of stochastic finite elements,” *Advances in Engineering Software*, vol. 31, no. 8–9, pp. 607–616, Aug. 2000, doi: 10.1016/S0965-9978(00)00034-X.

[7] F. Zangeneh-Nejad, D. L. Sounas, A. Alù, and R. Fleury, “Analogue computing with metamaterials,” *Nat Rev Mater*, vol. 6, no. 3, pp. 207–225, Oct. 2020, doi: 10.1038/s41578-020-00243-2.

[8] A. Silva, F. Monticone, G. Castaldi, V. Galdi, A. Alù, and N. Engheta, “Performing Mathematical Operations with Metamaterials,” *Science (1979)*, vol. 343, no. 6167, pp. 160–163, Jan. 2014, doi: 10.1126/science.1242818.

[9] H. Rajabalipanah, A. Momeni, M. Rahmanzadeh, A. Abdolali, and R. Fleury, “Parallel wave-based analog computing using metagratings,” *Nanophotonics*, vol. 11, no. 8, pp. 1561–1571, May 2022, doi: 10.1515/nanoph-2021-0710.

[10] T. Knightley, A. Yakovlev, and V. Pacheco-Peña, “Neural Network Design of Multilayer Metamaterial for Temporal Differentiation,” *Adv Opt Mater*, vol. 11, no. 5, p. 2202351, Mar. 2023, doi: 10.1002/adom.202202351.

[11] R. G. MacDonald, A. Yakovlev, and V. Pacheco-Peña, “Time derivatives via interconnected waveguides,” *Sci Rep*, vol. 13, no. 1, p. 13126, Aug. 2023, doi: 10.1038/s41598-023-40046-3.

[12] J. Sol, D. R. Smith, and P. del Hougne, “Meta-programmable analog differentiator,” *Nat Commun*, vol. 13, no. 1, p. 1713, Mar. 2022, doi: 10.1038/s41467-022-29354-w.

[13] E. A. Bezus, L. L. Doskolovich, D. A. Bykov, and V. A. Soifer, “Spatial integration and differentiation of optical beams in a slab waveguide by a dielectric ridge supporting high-Q resonances,” *Opt Express*, vol. 26, no. 19, p. 25156, Sep. 2018, doi: 10.1364/OE.26.025156.

[14] M. Park and Y. Park, “Metasurface-Based Imagers Enabled Arbitrary Optical Convolution Processing,” *Light Sci Appl*, vol. 11, no. 1, pp. 10–11, 2022, doi: 10.1038/s41377-022-00792-x.

[15] K. Wu, C. Soci, P. P. Shum, and N. I. Zheludev, “Computing matrix inversion with optical

networks," *Opt Express*, vol. 22, no. 1, p. 295, Jan. 2014, doi: 10.1364/OE.22.000295.
[16] N. Mohammadi Estakhri, B. Edwards, and N. Engheta, "Inverse-designed metastructures that solve equations," *Science (1979)*, vol. 363, no. 6433, pp. 1333–1338, Mar. 2019, doi: 10.1126/science.aaw2498.
[17] D. C. Tzarouchis, M. J. Mencagli, B. Edwards, and N. Engheta, "Mathematical operations and equation solving with reconfigurable metadevices," *Light Sci Appl*, vol. 11, no. 1, p. 263, Sep. 2022, doi: 10.1038/s41377-022-00950-1.
[18] S. Sun *et al.*, "Induced homomorphism: Kirchhoff's law in photonics," *Nanophotonics*, vol. 10, no. 6, pp. 1711–1721, Apr. 2021, doi: 10.1515/nanoph-2020-0655.
[19] M. Miscuglio *et al.*, "Approximate analog computing with metatronic circuits," *Commun Phys*, vol. 4, no. 1, p. 196, Dec. 2021, doi: 10.1038/s42005-021-00683-4.
[20] R. G. MacDonald, A. Yakovlev, and V. Pacheco-Peña, "Solving partial differential equations with waveguide-based metatronic networks," *Advanced Photonics Nexus*, vol. 3, no. 05, Oct. 2024, doi: 10.1117/1.APN.3.5.056007.
[21] A. Ventisei, A. Yakovlev, and V. Pacheco-Peña, "Exploiting Petri Nets for Graphical Modelling of Electromagnetic Pulse Switching Operations," *Adv Theory Simul*, vol. 2100429, p. 2100429, Dec. 2021, doi: 10.1002/adts.202100429.
[22] W. Rogers, C. Johnson-Richards, A. Yakovlev, and V. Pacheco-Peña, "Perfect splitting in rectangular-waveguide junctions for analog computing," *Phys Rev Appl*, vol. 21, no. 5, p. 054054, May 2024, doi: 10.1103/PhysRevApplied.21.054054.
[23] C. Johnson-Richards, A. Yakovlev, and V. Pacheco-Peña, "Topological Valley Photonic Waveguides: Scattering matrix evaluation for linear computing," Dec. 2024, [Online]. Available: http://arxiv.org/abs/2412.02388
[24] R. G. MacDonald, A. Yakovlev, and V. Pacheco-Peña, "Amplitude-Controlled Electromagnetic Pulse Switching Using Waveguide Junctions for High-Speed Computing Processes," *Advanced Intelligent Systems*, vol. 2200137, p. 2200137, Nov. 2022, doi: 10.1002/aisy.202200137.
[25] V. Pacheco-Peña and A. Yakovlev, "Computing with Square Electromagnetic Pulses," in *Handbook of Unconventional Computing,* 1st ed., A. Adamatzky, Ed., London: World Scientific, 2021, pp. 465–492. doi: 10.1142/9789811235740_0016.
[26] Y. D. Save, H. Narayanan, and S. B. Patkar, "Solution of Partial Differential Equations by electrical analogy," *J Comput Sci*, vol. 2, no. 1, pp. 18–30, Mar. 2011, doi: 10.1016/j.jocs.2010.12.006.
[27] G. Kron, "Electric circuit models of partial differential equations," *Electrical Engineering*, vol. 67, no. 7, pp. 672–684, Jul. 1948, doi: 10.1109/EE.1948.6444220.
[28] N. Engheta, "Circuits with Light at Nanoscales: Optical Nanocircuits Inspired by Metamaterials," *Science (1979)*, vol. 317, no. 5845, pp. 1698–1702, Sep. 2007, doi: 10.1126/science.1133268.
[29] Y. Li, I. Liberal, and N. Engheta, "Dispersion synthesis with multi-ordered metatronic filters," *Opt Express*, vol. 25, no. 3, p. 1937, Feb. 2017, doi: 10.1364/OE.25.001937.
[30] H. Caglayan, S.-H. Hong, B. Edwards, C. R. Kagan, and N. Engheta, "Near-Infrared Metatronic Nanocircuits by Design," *Phys Rev Lett*, vol. 111, no. 7, p. 073904, Aug. 2013, doi: 10.1103/PhysRevLett.111.073904.
[31] Y. Li and N. Engheta, "Microwave analogues of multi-ordered metatronic filters with waveguide metamaterials," in *2016 URSI Asia-Pacific Radio Science Conference (URSI AP-RASC)*, IEEE, Aug. 2016, pp. 530–533. doi: 10.1109/URSIAP-RASC.2016.7601273.
[32] N. Liu *et al.*, "Individual Nanoantennas Loaded with Three-Dimensional Optical Nanocircuits," *Nano Lett*, vol. 13, no. 1, pp. 142–147, Jan. 2013, doi: 10.1021/nl303689c.
[33] A. R. Rashed *et al.*, "Highly-Sensitive Refractive Index Sensing by Near-infrared Metatronic Nanocircuits," *Sci Rep*, vol. 8, no. 1, p. 11457, Jul. 2018, doi: 10.1038/s41598-018-29623-z.
[34] Y. Li, I. Liberal, and N. Engheta, "Metatronic analogues of the Wheatstone bridge," *Journal of the Optical Society of America B*, vol. 33, no. 2, p. A72, Feb. 2016, doi: 10.1364/JOSAB.33.000A72.
[35] W. Sun, X. Qin, H. Li, Z. Zhou, and Y. Li, "Impedance matching via ultrathin metatronic layer assisted by Smith Chart.," *Opt Express*, vol. 30, no. 14, pp. 25567–25580, Jul. 2022, doi: https://doi.org/10.1364/OE.465192.
[36] L. Burstein, *PDE Toolbox Primer for Engineering Applications with MATLAB® Basics*. Boca Raton: CRC Press, 2022. doi: 10.1201/9781003200352.