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Laplacian Surrogate of the EFIE based on Differential Forms: Application to Preconditioning

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Abstract

This work introduces a differential forms-based framework for constructing a novel surrogate for the electric field integral operator (EFIO) from low to moderate frequencies. The proposed methodology leverages a Hodge decomposition to introduce exterior calculus projectors, which are subsequently used to build a spectral equivalent of the EFIO. The discretization of the surrogate is achieved following discrete de Rham cohomology, and its potential is demonstrated by employing it to precondition the EFIO. Numerical results validate the effectiveness of the surrogate as a preconditioner, particularly in addressing ill-conditioning issues at low frequencies and for dense discretizations.

1 Introduction

Integral equations are essential in computational electromagnetics, particularly for modeling scattering from perfectly electrically conducting (PEC) objects. The electric field integral equation (EFIE) is widely used for its flexibility and accuracy. However, it suffers from severe ill-conditioning in critical scenarios, such as at low frequencies or under h-refinement, when discretization becomes denser. These challenges significantly hinder the performance of iterative solvers.

Various remedies have been proposed to address these issues. Calderón projector techniques [1], while effective, require Buffa-Christiansen (BC) functions based on dual grids, which are expensive to build. In [4], a refinement-free preconditioner of the EFIE using on quasi-Helmholtz projectors and a mixed Gram matrix is proposed. While this work provides a remedy to the EFIE conditioning issues, it does not provide a surrogate of the EFIE, which is highly suitable when performing a spectral analysis.

Here, we reformulate the EFIE using local operators within the language of differential forms. This approach leverages Hodge theory to construct a refinement-free surrogate operator that accurately reproduces the spectral properties of the EFIE from low to moderate frequencies without requiring mixed Gram matrices. The surrogate is designed to facilitate spectral analysis and serve as a preconditioner, relying on operations exclusively with sparse matrices, thus ensuring computational efficiency.

The paper is organized as follows: we first introduce fundamental concepts in differential forms and how they relate to vector calculus. Next, we describe the construction of the surrogate, including its derivation from Hodge theory and discretization via discrete de Rham sequences. We then demonstrate the application of the surrogate as a preconditioner for the EFIE and validate its performance through numerical experiments.

2 Notation and Background

Let Ω represent a simply connected perfect electric conductor (PEC) scatterer with a smooth boundary Γ, illuminated by a time-harmonic incident electric field $\boldsymbol{E}_i$, and surrounded by a homogeneous medium characterized by permittivity ε and permeability μ. The surface Γ is oriented by its outward pointing normal $\hat{\boldsymbol{n}}(\boldsymbol{r})$ at $\boldsymbol{r}$. We begin by introducing the foundational concepts of differential forms, which provide a powerful mathematical framework for expressing the EFIO in the language of differential forms. We denote $f^{(p)}$ as a differential p-form defined on Γ, where $0 \leq p \leq 2$. The Hodge star operator, $\star$, is defined as a mapping that transforms a p-form into a $(2-p)$-form. Using the Hodge star operator, we define an inner product between two forms, denoted as $\langle f, g \rangle = \int_\Gamma f \wedge \star g$ where $\wedge$ represents the wedge product, a bilinear operation on differential forms. In addition to the Hodge star, two essential differential operators are defined: the exterior derivative d, which maps a p-form to a $(p+1)$-form, and the coderivative operator δ, which maps a p-form to a $(p-1)$-form. The coderivative is related to the exterior derivative through the Hodge star as $\delta = \star d \star$. These operators satisfy the properties $d^2 = 0$ and $\delta^2 = 0$. Using these operators, a Hodge decomposition can be performed on $f^{(p)}$, by separating it into exact and coexact components as follows

$$f^{(p)} = d\alpha^{(p-1)} + \delta\beta^{(p+1)}, \tag{1}$$

Note that there is no harmonic component since a simply connected geometry is assumed. Furthermore, the de Rham Laplacian Δ is defined as $\Delta = d\delta + \delta d$, and can be applied to any p-form. Subsequently, we formulate the EFIO in the language of differential forms as follows

$$T = \mathrm{i}kT_\mathrm{s} - \frac{1}{\mathrm{i}k}T_\mathrm{h}, \tag{2}$$

where the singular and hypersingular operators are defined as $T_\mathrm{s} = V$ and $T_\mathrm{h} = \delta V d$, with V being the differential forms single-layer operator (SLO) defined in [3]. For the sake of conciseness, the detailed definition of this operator is not recalled here. The classical vector calculus version of the EFIE can be retrieved by applying the translation isomorphisms defined in [3, 8]. These isomorphisms bridge the Sobolev spaces to the differential forms setting [3], which yields the classical calculus EFIO [4]

$$\mathcal{T} = \mathrm{i}k\mathcal{T}_s - \frac{1}{\mathrm{i}k}\hat{\boldsymbol{n}} \times \nabla_\Gamma \mathcal{V}(\nabla_\Gamma \cdot), \tag{3}$$

where $\mathcal{V}$ is the SL operator in this formalism. The EFIO relates the current density $\boldsymbol{j}$ and $\boldsymbol{E}_i$ in the EFIE as $\mathcal{T}\boldsymbol{j} = -\hat{\boldsymbol{n}} \times \boldsymbol{E}_i$.

Subsequently, we move back to the differential forms formalism and introduce the discretization of the operators $\star$, d, δ, and Δ. The geometry Γ is decomposed into a mesh with p-simplices $e^{(p)}$ where the $p = 0$, 1, and 2 simplices are the N_v nodes, N_e edges, and N_f faces of the mesh, respectively. A differential form $f^{(p)}$ can then be discretized on the mesh such that its entries are defined as $[\boldsymbol{f}_p]_i = \int_{e_i^{(p)}} f^{(p)}$. Next, we proceed to discretize the operators d and δ. The discrete version of d is obtained by constructing the incidence matrix $\boldsymbol{C}_p$ [8], which relates the degrees of freedom (DoFs) of $f^{(p-1)}$ and $h^{(p)} = d f^{(p-1)}$ as

$$[\boldsymbol{h}_p]_m = \sum_n [\boldsymbol{C}_p]_{mn} [\boldsymbol{f}_{p-1}]_n. \tag{4}$$

For the discretization of $\star$, we adopt a discretization scheme based on surface Whitney interpolants [7]. This approach results in the Galerkin discrete Hodge operator, which acts on discrete p-forms and is defined as [5]

$$[\star_p]_{mn} = \langle w_m^{(p)}, w_n^{(2-p)} \rangle, \tag{5}$$

where $w_m^{(p)}$ are the Whitney p-interpolants [7] such that $f^{(p)} = \sum_i [\boldsymbol{f}_p]_i w_i^{(p)}$. This discrete operator allows us to relate the DoFs of discrete differential forms, such as through

$$\boldsymbol{g}_{p-2} = \star_p \boldsymbol{f}_p, \tag{6}$$

in which $g^{(p-2)} = \sum_i [\boldsymbol{g}_{p-2}]_i \tilde{w}_i^{(p-2)}$ and $\tilde{w}_i^{(p-2)}$ are dual Whitney forms that do not require to be defined explicitly in this work. To align with the notation commonly used for the discretization of the EFIE [2, 4], we introduce the discrete operators $\boldsymbol{\Lambda} = \boldsymbol{C_1} \in \mathbb{R}^{N_e \times N_v}$, $\boldsymbol{\Sigma} = \boldsymbol{C_2^{\mathrm{T}}} \in \mathbb{R}^{N_e \times N_f}$, $\boldsymbol{G}_\lambda = \star_0 \in \mathbb{R}^{N_v \times N_v}$, $\boldsymbol{G} = \star_1 \in \mathbb{R}^{N_e \times N_e}$, and $\boldsymbol{G}_p = \star_2 \in \mathbb{R}^{N_f \times N_f}$. Subsequently, using Fig. 1, which illustrates the discrete de Rham cohomology [7, 8], a natural discrete expression can be obtained for the de Rham Laplacian when applied to 0-forms and 2-forms

$$\boldsymbol{\Delta}_\Lambda = \boldsymbol{G}_\lambda^{-1} \boldsymbol{\Lambda}^{\mathrm{T}} \boldsymbol{G} \boldsymbol{\Lambda}, \tag{7}$$

$$\boldsymbol{\Delta}_\Sigma = \boldsymbol{\Sigma}^{\mathrm{T}} \boldsymbol{G}^{-1} \boldsymbol{\Sigma} \boldsymbol{G}_p, \tag{8}$$

respectively. Finally, the EFIE (3) (or equivalently (2)) is discretized with Rao-Wilton-Glisson (RWG) basis functions (or equivalently Whitney 1-forms) following the procedure described in [2], which results in the following linear system

$$\boldsymbol{T}\boldsymbol{J} = \boldsymbol{v}. \tag{9}$$

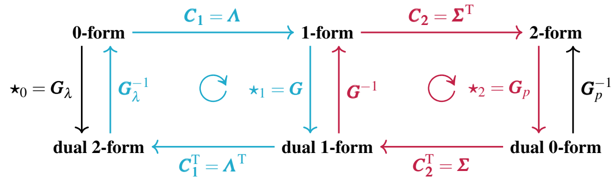

Figure 1. Discrete de Rham cohomology. The Hodge duality is represented by the Galerkin Hodge stars $\star_p$. The construction of the discrete Laplacian $\boldsymbol{\Delta}_\Lambda$ and $\boldsymbol{\Delta}_\Sigma$ follows the arrows colored in blue and red, respectively.

3 Methods and Applications

In this section, we introduce a surrogate framework for the EFIE that mimics its conditioning properties in critical regimes, such as h-refinement and low-frequency limits. This framework simplifies the study of the original formulation by using only local operators. We begin by deriving projectors from the Hodge decomposition, which form the basis for constructing the surrogate operator within the framework of differential forms while ensuring spectral equivalence with the standard EFIE. Subsequently, we provide a discretized version of the surrogate framework and demonstrate its application as a preconditioner.

3.1 Hodge Decomposition and Continuous Projectors

The starting point for the surrogate framework is the Hodge decomposition in (1). By applying to the p-form $j^{(p)}$, we derive the following relationships

$$d\, j^{(p)} = d\delta\beta^{(p+1)}, \tag{10}$$

$$\delta\, j^{(p)} = \delta d\alpha^{(p-1)}, \tag{11}$$

where $\alpha^{(p-1)}$ and $\beta^{(p+1)}$ are $(p-1)$- and $(p+1)$-forms, respectively. Solving for $\beta^{(p+1)}$ and $\alpha^{(p-1)}$, we obtain

$$\beta^{(p+1)} = (d\delta)^{-1}\, d\, j^{(p)}, \tag{12}$$

$$\alpha^{(p-1)} = (\delta d)^{-1}\, \delta\, j^{(p)}. \tag{13}$$

which can further be expressed in terms of p-forms as

$$\beta_2^{(p)} = \delta\,(d\delta)^{-1}\, d\, j^{(p)}, \tag{14}$$

$$\alpha_2^{(p)} = d\,(\delta d)^{-1}\, \delta\, j^{(p)}, \tag{15}$$

Substituting (14) and (15) in (1) yields $j^{(p)} = (P_\Lambda + P_\Sigma)\, j^{(p)}$, where the projection operators P_Λ and P_Σ are defined as

$$P_\Lambda = d(\delta d)^{-1}\delta, \tag{16}$$

$$P_\Sigma = \delta(d\delta)^{-1}d, \tag{17}$$

and satisfy the orthogonality and completeness relations

$$P_\Lambda P_\Sigma = 0, \quad P_\Sigma P_\Lambda = 0, \quad P_\Lambda + P_\Sigma = I. \tag{18}$$

3.2 Derivation of the Surrogate

Using the identity [6]

$$\mathcal{V}\nabla_\Gamma \cdot \nabla_\Gamma \mathcal{V} = \frac{\mathcal{I}}{4} + \mathcal{C}, \tag{19}$$

with $\mathcal{C}$ being a compact operator, and the translational isomorphisms introduced in [3], we obtain the following spectral identity in the language of differential forms

$$V(d\delta)V = \frac{I}{4} + C, \tag{20}$$

where C is obtained from $\mathcal{C}$. We can thus approximate V by its spectrally equivalent operator

$$S = \frac{1}{2}(d\delta)^{-\frac{1}{2}}. \tag{21}$$

By observing that $(\delta S d)^2 = \frac{1}{4}\delta d$ and using (18), we derive an additional identity

$$\delta S d = \frac{1}{2}(\delta d)^{\frac{1}{2}}. \tag{22}$$

A spectrally equivalent operator for the EFIO (2) then takes the form

$$\tilde{T} = \mathrm{i}kS - \frac{1}{\mathrm{i}k}\delta S d. \tag{23}$$

Next, to allow discretization with de Rham cohomology, (23) will be further simplified by applying a perturbed Hodge decomposition for $k \to 0$. Defining $P_L = \left[\frac{1}{\mathrm{i}k}\delta,\, d\right]^{\mathrm{T}}$ and $P_R = [d,\, \mathrm{i}k\delta]$ and using (18), we obtain at low frequencies ($k \to 0$)

$$\left(P_L \tilde{T} P_R\right)_{k\to 0} = \frac{1}{2}\begin{bmatrix}(\delta d)^{\frac{1}{2}} & 0 \\ 0 & (d\delta)^{\frac{3}{2}}\end{bmatrix}. \tag{24}$$

A surrogate $\tilde{V}$ of the original operator can then be retrieved for $k \to 0$

$$\begin{aligned}\tilde{V} &= P_L^{-1}\left(P_L \tilde{T} P_R\right)_{k\to 0} P_R^{-1} \\ &= \frac{1}{2}\mathrm{i}kd(\delta d)^{-3/2}\delta - \frac{1}{2\mathrm{i}k}\delta(d\delta)^{-1/2}d,\end{aligned} \tag{25}$$

where the inverses $P_L^{-1} = \left[\mathrm{i}kd(\delta d)^{-1},\, \delta(d\delta)^{-1}\right]^{\mathrm{T}}$ and $P_R^{-1} = \left[(\delta d)^{-1}\delta,\, 1/(\mathrm{i}k)(d\delta)^{-1}d\right]$ are derived using (18). The surrogate operator in (25) can then inverted

$$\tilde{V}^{-1} = \frac{2}{\mathrm{i}k}d(\delta d)^{-\frac{1}{2}}\delta + 2\mathrm{i}k\delta(d\delta)^{-\frac{3}{2}}d, \tag{26}$$

such as $\tilde{V}^{-1}\tilde{V} = I$. The next step is to discretize the surrogate operator by using Fig. 1 and using the fact that the EFIO given in (3) follows the mapping $H_{\mathrm{div}}(\Gamma) \to H_{\mathrm{div}}(\Gamma)$ [4], which means that the sequence in Fig. 1 should start at the 1-form and end also at this point. This results in the following discrete surrogate

$$\tilde{\boldsymbol{V}} = \frac{\mathrm{i}k}{2}\boldsymbol{\Lambda}\boldsymbol{\Delta}_\Lambda^{-\frac{3}{2}}\boldsymbol{G}_\lambda^{-1}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G} - \frac{1}{2\mathrm{i}k}\boldsymbol{G}^{-1}\boldsymbol{\Sigma}\boldsymbol{G}_p\boldsymbol{\Delta}_\Sigma^{-\frac{1}{2}}\boldsymbol{\Sigma}^{\mathrm{T}}, \tag{27}$$

having the following inverse

$$\tilde{\boldsymbol{V}}^{-1} = \frac{2}{\mathrm{i}k}\boldsymbol{\Lambda}\boldsymbol{\Delta}_\Lambda^{-\frac{1}{2}}\boldsymbol{G}_\lambda^{-1}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G} + 2\mathrm{i}k\boldsymbol{G}^{-1}\boldsymbol{\Sigma}\boldsymbol{G}_p\boldsymbol{\Delta}_\Sigma^{-\frac{3}{2}}\boldsymbol{\Sigma}^{\mathrm{T}}. \tag{28}$$

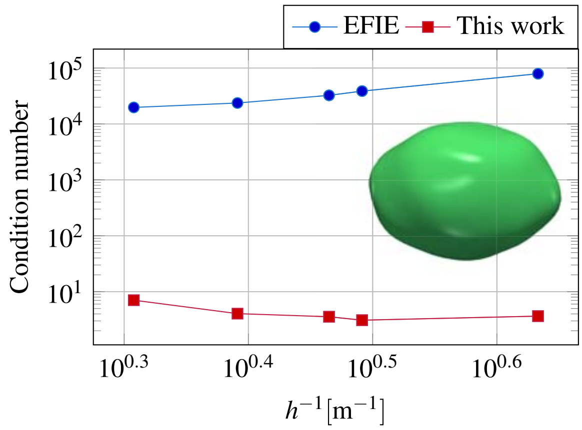

Figure 2. Condition number of the EFIE (9) and the preconditioned EFIE (29) as a function of discretization for $k = 0.1\,\mathrm{m}^{-1}$. The simulated structure is a smoothly deformed sphere.

3.3 Application to Preconditioning the EFIE

In the following, we employ the inverse of the discrete surrogate $\tilde{\boldsymbol{V}}^{-1}$ to precondition the discrete EFIE (9) at low frequencies and for dense discretization. To avoid the computation of Laplacian square roots when solving iteratively the system, which is computationally expensive, we symmetrize the preconditioned equation so that $\tilde{\boldsymbol{V}}^{-2}$ appears.

$$\boldsymbol{T}\tilde{\boldsymbol{V}}^{-2}\boldsymbol{T}\boldsymbol{J} = \boldsymbol{T}\tilde{\boldsymbol{V}}^{-2}\boldsymbol{v} \tag{29}$$

This preconditioning approach ensures well conditioning while maintaining compatibility with efficient computational implementations such as in [2]. Note that particular care should be taken when applying the $\tilde{\boldsymbol{V}}^{-1}$ to avoid loss of accuracy at very low frequencies. Due to finite machine precision, the terms that go to zero such as $\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Sigma}$ should be manually set to 0, as it is done in the quasi-Helmholtz EFIE [2].

4 Numerical Results

The surrogate's performance as a preconditioner is validated by demonstrating the spectral stability of the preconditioned EFIO in low-frequency scenarios and for dense discretizations. As shown in Fig. 2, the condition number of the EFIE (9) and the preconditioned EFIE (29) is shown as a function of discretization for a frequency of $k = 0.1\,\mathrm{m}^{-1}$. The results highlight the improved conditioning achieved with the preconditioned EFIE under high discretization levels. Additionally, Fig. 3 shows the condition number as a function of frequency, further demonstrating the effectiveness of the preconditioner in mitigating ill-conditioning at low frequencies. The simulated structure in both cases is a smoothly deformed sphere. Finally, a NASA almond model is illuminated by a $k = 0.1\,\mathrm{m}^{-1}$ x-polarized plane wave propagating along the z-direction. Figure 4 shows the history of the relative residual error of the EFIE (9) and the preconditioned EFIE (29) when using the conjugate gradient squared (CGS) iterative solver with a tolerance of 10^{-6}.

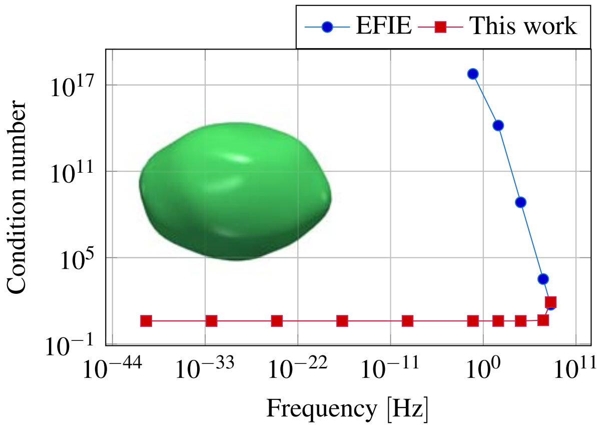

Figure 3. Condition number of the EFIE (9) and the preconditioned EFIE (29) as a function of frequency. The simulated structure is a smoothly deformed sphere.

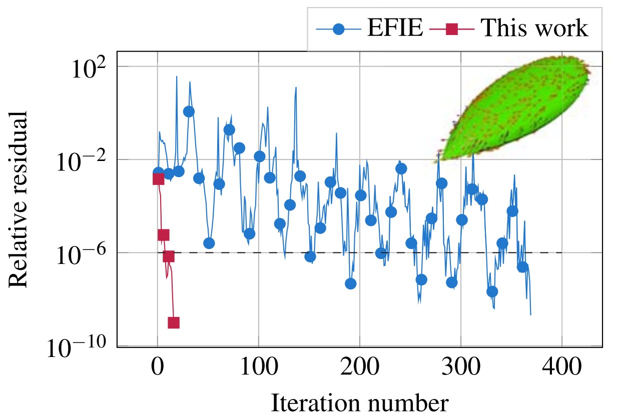

Figure 4. Relative residual errors of the EFIE (9) and preconditioned EFIE (29) solutions versus iterations with a plane wave excitation at $k = 0.1\,\mathrm{m}^{-1}$. The simulated structure is the NASA almond.

The well conditioning of the preconditioned EFIE with this challenging geometry translates into a decrease from 368 to 15 of the number of iterations.

5 Conclusion

We have proposed a differential forms-based approach for constructing a Laplacian surrogate for the EFIE and an associated preconditioner. The discrete surrogate mimics the spectral properties of the EFIE while its associated preconditioner addresses its low frequency and h-refinement ill-conditioning. Future work includes extending this framework to other integral equations.

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