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On the Existence of TE Waves in a Circular Dielectric Rod

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Abstract

This paper investigates the dispersion properties of symmetric TE surface waves in a homogeneous dielectric cylinder of circular cross section. Analytical methods, including the formulation and analysis of dispersion equation (DE) solution as implicit functions employing Cauchy problems, are used to prove the existence and determine propagation constants of these waves.

1 Introduction

Analysis of waves in open dielectric waveguides has always been a key issue not only of the electromagnetic field theory, but also in mathematical physics, differential equations, and operator spectral theory [1, 2, 3]. In fact, the corresponding boundary value problems (BVPs) for Maxwell and Helmholtz equations with piecewise constant coefficients that form the basis of mathematical models of the wave propagation are nonselfadjoint and the spectral parameter enters the conditions at infinity (as well as the transmission conditions) in a nonlinear manner. These circumstances mean that standard investigation techniques cannot be applied and require the necessity of elaborating new approaches and methods that have been known since the early 1990s as spectral theory of open structures [4]. Central points of the theory are: rigorous proofs of the existence of waves in open guides, description of the location of the wave spectra on the complex plane, and development of the solution and calculation techniques As applied to a canonical dielectric waveguide structure, a homogeneous dielectric rod (DR) of circular cross section, a rigorous proof of the existence of TM waves has been obtained [4, 5, 6] relatively recently leading to the creation of specific analytical and numerical approaches [7, 8]. Remarkably, similar proofs for TE waves in DR are still not available in the literature which dictate the accomplishment of the present research involving a complete BVP solution routine. We reduce the determination of the propagation constants of the DR TE waves to the solution of a multiparameter dispersion equation (DE) and prove the existence of the DE roots and develop a method for their computation using the theory of implicit functions of several complex variables.

The permittivity $\varepsilon = \widetilde{\varepsilon}\varepsilon_0$ of a homogeneous dielectric cylinder of circular cross section, a DR, is governed by the relation

$$\widetilde{\varepsilon} = \begin{cases} \varepsilon_l, & 0 \leqslant \rho < r_1, \\ \varepsilon_c, & \rho > r_1, \end{cases} \tag{1}$$

where $\varepsilon_l \geqslant 1, \varepsilon_c \geqslant 1$ are positive constants such that

$$\varepsilon_c < \varepsilon_l. \tag{2}$$

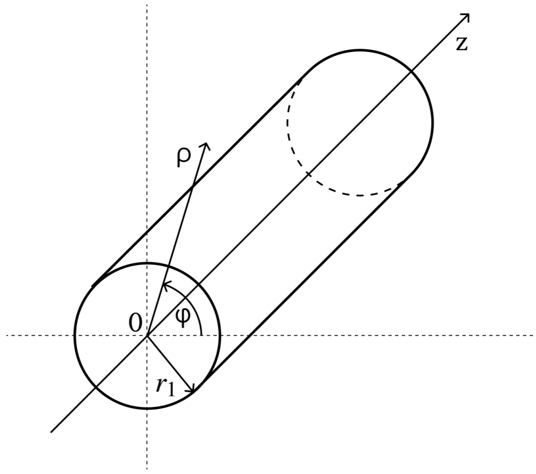

Figure 1. Dielectric rod.

The symmetric (azimuthally-independent) TE waves in the polar coordinates have the nonzero components

$$\mathbf{E} = [0, E_2(\rho, z), 0], \quad \mathbf{H} = [H_1(\rho, z), 0, H_3(\rho, z)], \tag{3}$$

where

$$E_2 = u(\rho)e^{i\gamma z}, \tag{4}$$

γ is the wave propagation constant (spectral parameter), and potential function u is a cylindrical function satisfying the Bessel equation of index 1, and is sought in the form

$$u(\rho) = \begin{cases} c_1 J_1(k_l \rho), & 0 \leqslant \rho < r_1, \\ c_3 K_1(k_c \rho), & \rho > r_1, \end{cases} \tag{5}$$

where c_1 and c_3 are arbitrary nonzero constants, J_1 and K_1 are, respectively, the Bessel and McDonald cylindrical function of index 1, $k_0^2 = \omega^2 \mu \varepsilon_0$, $k_0 = \frac{\omega}{c} = \frac{2\pi}{\Lambda}$ is the wavenumber of vacuum, and

$$k_l^2 := k_0^2 \varepsilon_l - \gamma^2, \;\; k_c^2 := \gamma^2 - k_0^2 \varepsilon_c. \tag{6}$$

The form of the solution (5) at $\rho > r_1$ is governed by an asymptotic representation at large $|z|$

$$K_1(z) = C_{K_1} \frac{e^{-z}}{\sqrt{z}} \left[1 + O\left(\frac{1}{|z|}\right)\right] \tag{7}$$

which provides the fulfillment of condition at infinity

$$u(\rho) \to 0, \quad \rho \to \infty \tag{8}$$

specifying symmetric *surface* waves in DR described in terms of real-valued quantities and defined as real-valued functions of a real variable γ or $\lambda = \gamma^2$ varying on a certain interval I.

2 Dispersion equations

2.1 Dispersion equations for TE waves in a homogeneous dielectric cylinder of circular cross section (dielectric rod)

To obtain the desired DE apply to (5) the transmission conditions on the external boundary of DR

$$\begin{aligned} u|_{\rho=r_1-0} - u|_{\rho=r_1+0} = 0, \\ u'\big|_{\rho=r_1-0} - u'\big|_{\rho=r_1+0} = 0 \end{aligned} \tag{9}$$

Make use of the relations of the type

$$\begin{aligned} u'(\rho) = \frac{du}{d\rho} = \frac{d}{d\rho} J_1(k_l\rho) = k_l J_1'(k_l\rho) = \\ = k_l J_0(k_l\rho) - \frac{1}{\rho} J_1(k_l\rho), \end{aligned} \tag{10}$$

to get

$$\begin{aligned} c_1 J_1(k_l r_1) = c_3 K_1(k_c r_1), \\ c_1 k_l J_1'(k_l r_1) - c_3 k_c K_1'(k_c r_1) = 0, \end{aligned} \tag{11}$$

which yields

$$k_l \frac{J_1'(k_l r_1)}{J_1(k_l r_1)} - k_c \frac{K_1'(k_c r_1)}{K_1(k_c r_1)} = 0. \tag{12}$$

Using the recurrence relations [9] for cylindrical functions and (10) rewrite (12) as

$$k_l \frac{J_0(k_l r_1)}{J_1(k_l r_1)} + k_c \frac{K_0(k_c r_1)}{K_1(k_c r_1)} = 0, \tag{13}$$

which is the desired DE.

Taking $\varepsilon_c = 1$ (surrounding medium for $\rho > r_1$ is vacuum), setting $\varepsilon = \varepsilon_l > 1$, introducing the dimensionless variables

$$\begin{aligned} \hat{\gamma} = \frac{\gamma}{k_0},\ \kappa = k_0 r_1,\ x = \kappa\sqrt{\varepsilon - \hat{\gamma}^2},\ u = \kappa\sqrt{\varepsilon - 1}, \\ w = \sqrt{u^2 - x^2} = \kappa\sqrt{\hat{\gamma}^2 - 1}, \end{aligned} \tag{14}$$

so that

$$k_l^2 = k_0^2(\varepsilon - \hat{\gamma}^2) = \frac{x^2}{r_1^2}, \quad k_c^2 = k_0^2(\hat{\gamma}^2 - 1) = \frac{w^2}{r_1^2}, \tag{15}$$

and multiplying both sides by r_1, use variables from (14) and rewrite DE (13) in the form

$$x\frac{J_0(x)}{J_1(x)} + w\frac{K_0(w)}{K_1(w)} = 0, \tag{16}$$

which is the desired DE for TE waves in DR.

2.2 Analysis of the solvability of dispersion equation for TE waves

Recast (16) to the form

$$F_R(x) = G_R(x), \tag{17}$$

or

$$\Phi_D(x) = 0, \quad \Phi_D(x) \equiv F_R(x) - G_R(x), \tag{18}$$

where

$$\begin{aligned} F_R(x) = x\,\mathrm{cotj}(x), \quad \mathrm{cotj}(x) := \frac{J_0(x)}{J_1(x)}, \\ G_R(x) = -\sqrt{u^2 - x^2}\frac{K_0(\sqrt{u^2 - x^2})}{K_1(\sqrt{u^2 - x^2})}. \end{aligned} \tag{19}$$

For fixed $\varepsilon > 1$ and $\kappa > 0$, DE (16) may be considered in the form (17) w. r. t. the new variable (instead of γ)

$$x = \kappa\sqrt{(\varepsilon - 1) - w^2} = \kappa\sqrt{\varepsilon - \gamma^2}. \tag{20}$$

We have $x \in (0, \kappa\sqrt{\varepsilon - 1})$,

$$x = u \quad \text{at} \quad \gamma = \pm 1. \tag{21}$$

Relation (21) is used below to determine explicitly roots of DE (17) or, equivalently, zeros of the function $\Phi_D(x)$ defined by (18).

The existence and location of the real symmetric surface TE wave spectra can be proved, following [5], by verifying the occurrence of (real positive) roots of DE (16) or (17) (i.e. zeros of of function $\Phi_D(x)$ defined by (18)) and considering, for fixed $\varepsilon > 1$ and $\kappa > 0$, equations (17) and (18) w.r.t. the (real) independent variable x given by (20).

If $x^* \in (0, \kappa\sqrt{\varepsilon - 1})$ is a root of (18), then

$$\hat{\gamma}^* = \sqrt{\varepsilon - \left(\frac{x^*}{\kappa}\right)^2} \in (1, \sqrt{\varepsilon}) \tag{22}$$

is the propagation constants of a real symmetric surface TE wave. For real surface TE waves in DR the sought roots are located on the interval $(0, u)$ which corresponds to an interval $\hat{\gamma} \in (1, n)$, $n = \sqrt{\varepsilon} > 1$.

The results of the analysis regarding the occurrence of real roots of the DE can be summarized in the form of a concise statement:

Denote by ν_m^n the mth zeros of Bessel functions $J_n(x)$ ($n = 0,1$) alternating according to $\nu_m^0 < \nu_m^1 < \nu_{m+1}^0 < \nu_{m+1}^1$, $m = 1,2,\ldots$ Then the function

$$\tilde{\Phi}_D(\hat{\gamma}) = x\frac{J_0(x)}{J_1(x)} + w\frac{K_0(w)}{K_1(w)},\quad x = \kappa\sqrt{\varepsilon - \hat{\gamma}^2},\ \ w = \kappa\sqrt{\hat{\gamma}^2 - 1}, \tag{23}$$

with $\kappa > 0$ and $\varepsilon > 1$ has

- no real positive roots (or zeros) for $\hat{\gamma} \in (1, \sqrt{\varepsilon})$ (or for $x \in (0,u)$, $u = \kappa\sqrt{\varepsilon - 1}$) if

$$u = \kappa\sqrt{\varepsilon - 1} < \nu_1^0; \tag{24}$$

- $N \geq 1$ real positive roots (or zeros) on the interval $\hat{\gamma} \in [1, \sqrt{\varepsilon})$ (or $x \in (0,u]$ if

$$u = \kappa\sqrt{\varepsilon - 1} \in [\nu_N^0, \nu_N^1),\quad J_m(\nu_N^m) = 0,\ \ m = 0,1,\ \ N = 1,2,\ldots. \tag{25}$$

3 Determination of roots of dispersion equations. Existence of the real surface TE waves

3.1 Roots as implicit functions

DE (16) (equivalently, (17) or (23)) with $\kappa > 0$ and $\varepsilon > 1$ (and the function $\tilde{\Phi}_D(\hat{\gamma}) = \Phi_D(x), x = \kappa\sqrt{\varepsilon - \hat{\gamma}^2}$ defined by (18)) can be represented in an equivalent form

$$\tilde{\Phi}_I(\mathbf{P}) = 0,\quad \tilde{\Phi}_I(\mathbf{P}) = p\frac{J_0(\kappa p)}{J_1(\kappa p)} + q\frac{K_0(\kappa q)}{K_1(\kappa q)},\quad p = \sqrt{\varepsilon - \hat{\gamma}^2},\ \ q = \sqrt{\hat{\gamma}^2 - 1}, \tag{26}$$

and considered as a generalized DE (GDE) with $\tilde{\Phi}_D$ being a function of several complex or real variables forming the parameter vector $\mathbf{P} = (\mathbf{w}, \mathbf{z}) \in C^n$, $n = n_1 + n_2$, where C^n is an n-dimensional complex or real space, divided into the spectral (dependent) set of the sought quantities $\mathbf{w} = \left\{w_j\right\}_{j=1}^{n_1}$ and nonspectral (independent) variable set $\mathbf{z} = \left\{z_j\right\}_{j=1}^{n_2}$ (geometrical and material parameters).

The goal is to determine generalized dispersion curve (GDC) $\mathbf{w} = \mathbf{w}(\mathbf{z})$ as an implicit function defined by GDE (26).

In the considered case of the DR real TE waves, $\mathbf{w} = \hat{\gamma}$ is one (real) spectral parameter ($n_1 = 1$) and κ and ε are two ($n_2 = 2$) nonspectral parameters forming the vector $\mathbf{z} = (\kappa, \varepsilon)$.

3.2 Reduction to the solution of Cauchy problems

The sought implicit functions are governed by the equations and the initial conditions that follow from the 'cutoff' properties of $\tilde{\Phi}_D$:

$$\tilde{\Phi}_I(\hat{\gamma}, \kappa) = 0,\quad \hat{\gamma} = \hat{\gamma}(\kappa),\quad \hat{\gamma}(\kappa_{1,m}^1) = 1,\quad \kappa_{1,m}^1 = \frac{\nu_m^0}{\sqrt{\varepsilon - 1}},\ \ m = 1,2,\ldots, \tag{27}$$

for fixed $\varepsilon > 1$, or

$$\tilde{\Phi}_I(\hat{\gamma}, \varepsilon) = 0,\quad \hat{\gamma} = \hat{\gamma}(\varepsilon),\quad \hat{\gamma}(\varepsilon_{1,m}^1) = 1,\quad \varepsilon_{1,m}^1 = 1 + \left(\frac{\nu_m^0}{\kappa}\right)^2,\ \ m = 1,2,\ldots, \tag{28}$$

for fixed $\kappa > 0$.

The sought implicit functions $\hat{\gamma} = \hat{\gamma}_m(\kappa)$ or $\hat{\gamma} = \hat{\gamma}_m(\varepsilon)$ (the DE roots) are determined as solutions to the corresponding Cauchy problems

$$\begin{cases} \frac{d\hat{\gamma}_m}{d\kappa} = \Theta^{*\kappa}(\hat{\gamma}, \kappa), \\ \hat{\gamma}_m(\kappa = \kappa_{1,m}^1) = 1, \end{cases}\quad \Theta^{*\kappa}(\hat{\gamma}, \kappa) = -\frac{\frac{\partial \tilde{\Phi}_I}{\partial \kappa}}{\frac{\partial \tilde{\Phi}_I}{\partial \hat{\gamma}}}, \tag{29}$$

and

$$\begin{cases} \frac{d\hat{\gamma}_m}{d\varepsilon} = \Theta^{*\varepsilon}(\hat{\gamma}, \varepsilon), \\ \hat{\gamma}_m(\varepsilon = \varepsilon_{1,m}^1) = 1, \end{cases}\quad \Theta^{*\varepsilon}(\hat{\gamma}, \varepsilon) = -\frac{\frac{\partial \tilde{\Phi}_I}{\partial \varepsilon}}{\frac{\partial \tilde{\Phi}_I}{\partial \hat{\gamma}}}. \tag{30}$$

Index m labels the roots (implicit functions) $\hat{\gamma}_m(\kappa)$ and $\hat{\gamma}_m(\varepsilon)$ defined by the initial conditions in (29) and (30).

3.3 Existence of real surface TE waves in terms of unique solvability of Cauchy problems

The determination of the DE roots as implicit functions solving Cauchy problems (29) and (30) is validated and makes sense because the following solvability conditions are fulfilled which guarantee the existence of unique solution to each of (29) and (30) and the desired existence of the DR (real surface) TE waves:

(1) $\tilde{\Phi}_I(1, \kappa_{1,m}^1) = 0$ and $\tilde{\Phi}_I(1, \varepsilon_{1,m}^1) = 0$ (which is true according to the above analysis), and

(2) $\frac{\partial \tilde{\Phi}_I}{\partial \hat{\gamma}}(1, \kappa_{1,m}^1) \neq 0$ and $\frac{\partial \tilde{\Phi}_I}{\partial \hat{\gamma}}(1, \varepsilon_{1,m}^1) \neq 0$.

The Θ^* functions were calculated explicitly which enabled obtaining the sought implicit functions $\hat{\gamma}(\kappa)$ and $\hat{\gamma}(\varepsilon)$ by solving (29) and (30) numerically using the fourth-order Runge–Kutta method with accuracy control.

3.4 Numerical and wave classification

The graphs of GDCs, namely, of implicit functions $\hat{\gamma}_m(\kappa)$ for different indices m and values of ε are presented below.

Each GDC $\hat{\gamma}(\kappa) = \hat{\gamma}_m(\kappa)$ in Figs. 2 and 3 plots the unique solution of (29) and the corresponding root of DE (26) against varying κ for a fixed ε. The initial condition in (29) is uniquely governed by index m and the corresponding value of $\kappa = \kappa^1_{1,m}$ given by (27) and specifies a particular GDC $\hat{\gamma}_m(\kappa)$, so that m denotes the ordinal number of the $\gamma_m(\kappa)$ curve starting (according to the initial condition in (29)) at the value $\hat{\gamma}_m(\kappa) = 1$. Each solution of DE with the index $m = 1,2,\ldots$ in the initial condition in (29), i.e., each GDC, corresponds to a real surface wave of the type TE_m.

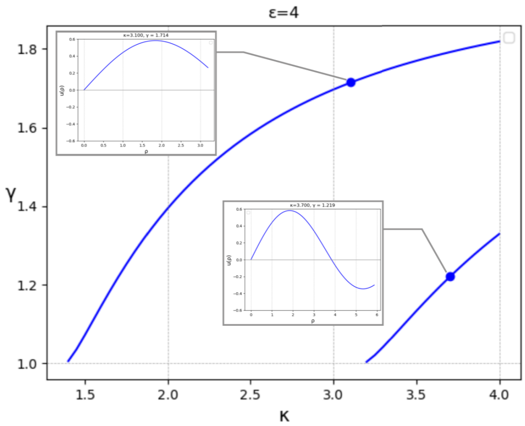

Figure 2. Plots of $\hat{\gamma}_m(\kappa), m = 1,2$, for $\varepsilon = 4$.

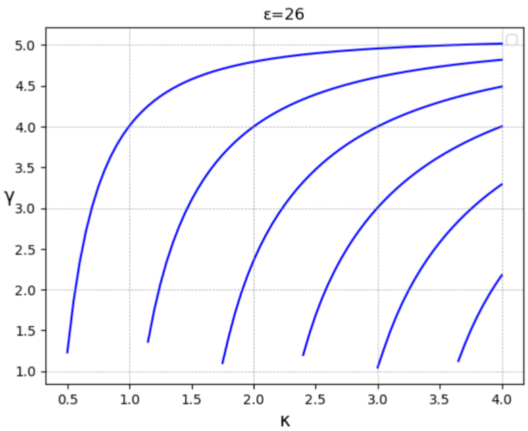

Figure 3. Plots of $\hat{\gamma}_m(\kappa), m = 1,2,\ldots,6$ for $\varepsilon = 26$.

Figure 2 demonstrates the potential functions $u(\rho), \rho \in [0; r_1)$, corresponding to the characteristic points on $\hat{\gamma}_m(\kappa)$. The number of oscillations of $u(\rho), \rho \in [0; r_1)$, is in relation with the index values $m = 1,2$ and specifies the TE wave type. Namely, it can be seen in Figure 2 that as index m increases, the number of oscillations of $u(\rho)$ on the interval of the radius variation inside the cross-sectional circle of the cylinder $\rho \in [0; r_1)$ increases too confirming the correspondence of the curves $\hat{\gamma}_m(\kappa)$ to TE_m-type waves for different values of ε and κ. Also, a comparison of Figures 2 and 3 shows that with an increase in ε, the number of curves $\gamma_w(\kappa)$ increases as well. This means that for higher parameter values, the maximum possible number of real waves $N = max(m)$ becomes greater, since for greater values of ε and κ, the condition

$$\begin{aligned} &u = \kappa\sqrt{\varepsilon - 1} \in [\nu^0_N, \nu^1_N), \\ &J_m(\nu^m_N) = 0, \quad m = 0,1, \quad N = 1,2,\ldots \end{aligned} \tag{31}$$

is satisfied with greater N so that DE has N real positive roots on the interval $\hat{\gamma} \in [1, \sqrt{\varepsilon})$, yielding the existence of N real TE_m-type waves.

4 Conclusion

We have verified the existence and location of the real TE wave spectra in a circular DR and set forth the analytical and computational techniques that employ determination of the parameter dependence of the wave propagation constants by calculating the corresponding implicit functions defined by DE as solutions to the Cauchy problems. The method can be generalized as a tool to determine all types of complex waves in DR.

References

[1] A.W. Snyder, J. Love, *Optical Waveguide Theory*. Boston: Kluwer, 1983.

[2] H.-G. Unger, *Planar Optical Waveguides and Fibres*. Oxford: Clarendon Press, 1977.

[3] Y. Shestopalov, Y. Smirnov, E. Smolkin, *Optical Waveguide Theory. Mathematical Models, Spectral Theory and Numerical Analysis*. Singapore: Springer, 2022.

[4] V. Shestopalov and Y. Shestopalov, *Spectral Theory and Excitation of Open Structures*. London: IEE, 1996.

[5] Y. Shestopalov, E. Kuzmina, and A. Samokhin. (2014, Sep-Oct.) "On a Mathematical Theory of Open Metal–Dielectric Waveguides." *FERMAT*. vol. 5. Available: https://www.e-fermat.org/files/articles/153bac073a019f.pdf

[6] Y. Shestopalov, "Complex Waves in a Dielectric Waveguide," *Wave Motion*, vol. 82, pp.16–19, Nov. 2018.

[7] Y. Shestopalov. (2021, May), "Resonance Scattering by a Circular Dielectric Cylinder." *Radio Science*, vol. 56, issue 6, pp. 172-178. Available: https://doi.org/10.1029/2020RS007095

[8] Y. Shestopalov. (2019, Jan), "Trigonometric and cylindrical polynomials and their applications in electromagnetics." *Applicable Analysis*, vol. 99, issue 16, pp. 2807-2822. Available: https://doi.org/10.1080/00036811.2019.1584290

[9] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1972.