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Ansatz for longitudinal solution for TEM-wave propagation through a metamaterial composite in a hollow waveguide

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Abstract

We derive and investigate the longitudinal field solution for the TEM mode inside a hollow waveguide with an arbitrarily shaped cross-section, filled with an impedance-matched metamaterial composite graded in the z-direction. The relative permittivity $\varepsilon(\omega,z)$ and permeability $\mu(\omega,z)$ are assumed to vary according to hyperbolic tangent functions. We obtain exact analytical solutions to Maxwell's equations for both lossless and lossy media. We introduce an analytical ansatz for the longitudinal field solution and compare it to the derived longitudinal field solution. An excellent agreement between the results from the analytical solution and the assumed analytical ansatz is obtained. This confirms that the proposed analytical ansatz for the longitudinal field solution fulfills the governing wave equation.

1 Introduction

Graded metamaterials are used in many state-of-the-art electromagnetic devices and designs, including invisibility cloaking [1], antennas [2], filters [3], and absorbers [4]. Despite the benefits of using metamaterials in many cost- and energy-efficient designs, there are still issues in the analysis and simulation of such materials. These issues include difficulty in developing accurate parameter extraction methods [5] and numerical simulation algorithms [6], while reducing their computation complexity to reasonable time frames. For this reason, analytical studies of generic metamaterial-based devices is beneficial as they provide a computationally efficient insight into the behavior of the electromagnetic systems. In the present work, we propose a method for analytically solving the wave equation for a waveguide structure with an arbitrarily shaped cross-section, filled with a metamaterial-based composite. The proposed method is significant because it can be utilized for analytically studying wave propagation and wave phenomena in waveguides filled with a variety of media with characteristics such as: periodicity, grading, and spatial- and frequency dependence; over any given frequency regime. Furthermore, we propose an analytical ansatz for the longitudinal field solution that is equivalent to the solution derived from the wave equation. Thus, providing a useful ansatz for studying both metamaterial-based applications and traditional applications involving regular materials.

2 Problem Description

We consider a hollow waveguide structure with an arbitrarily shaped cross-section. The left- and right-hand half of the waveguide is filled with a right-handed material (RHM) and a left-handed material (LHM) respectively. These materials are impedance-matched and possess a graded transition profile between each other. Fig. 1 shows the behavior of the real and imaginary parts of the permittivity $\varepsilon(\omega,z)$ and permeability $\mu(\omega,z)$ of the RHM-LHM composite along the length L of the waveguide, i.e. between $-L/2$ and $+L/2$.

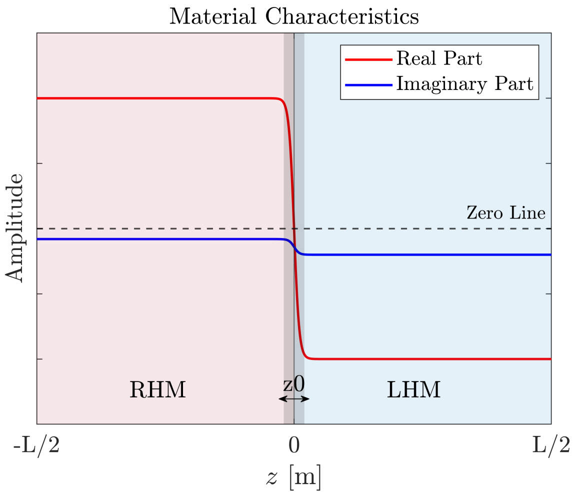


Figure 1. Graded RHM-LHM transition along the entire length L of the waveguide structure.

Using the $\exp(\mathrm{i}\omega t)$ time convention, the permittivity and permeability functions for the entire waveguide structure can be defined using the following continuous functions:

$$\varepsilon(\omega,z) = -\varepsilon_0 \frac{\varepsilon_{I1}+\varepsilon_{I2}}{2\beta}\left[\tanh\left(\frac{z}{z_0}\right)+\mathrm{i}\beta\right] \tag{1}$$

$$\mu(\omega,z) = -\mu_0 \frac{\mu_{I1}+\mu_{I2}}{2\beta}\left[\tanh\left(\frac{z}{z_0}\right)+\mathrm{i}\beta\right] \tag{2}$$

For the permittivity and permeability functions in (1) and (2) to fulfill the condition of constant wave impedance in the entire structure, we require that the constant β is defined as follows [7]:

$$\beta = \frac{\varepsilon_{I1}+\varepsilon_{I2}}{2\varepsilon_R - \mathrm{i}(\varepsilon_{I1}-\varepsilon_{I2})} = \frac{\mu_{I1}+\mu_{I2}}{2\mu_R - \mathrm{i}(\mu_{I1}-\mu_{I2})} \tag{3}$$

In the RHM-material $\varepsilon(\omega,z) = \varepsilon_0(+\varepsilon_R - \mathrm{i}\varepsilon_{I1})$ and $\mu(\omega,z) = \mu_0(+\mu_R - \mathrm{i}\mu_{I1})$ for $z \to -\infty$. In the LHM-material, we have that $\varepsilon(\omega,z) = \varepsilon_0(-\varepsilon_R - \mathrm{i}\varepsilon_{I2})$ and $\mu(\omega,z) = \mu_0(-\mu_R - \mathrm{i}\mu_{I2})$ for $z \to +\infty$. This asymptotic behavior describes two different impedance-matched passive materials in the waveguide. Based on the value of the loss factor β defined in (3), permittivity and permeability functions in (1) and (2) can describe both lossless- and lossy graded impedance-matched RHM-LHM composites, which we will consider separately. The present study will focus on Transverse ElectroMagnetic (TEM) wave propagation. For TEM waves propagating in an inhomogeneous media, the wave equation for the transverse electric field $\boldsymbol{E}(x,y,z)$ is defined as follows (TEM waves have $E_z = 0$):

$$\nabla^2\boldsymbol{E} - \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}z}\frac{\partial \boldsymbol{E}}{\partial z} + \omega^2\varepsilon\mu\boldsymbol{E} = 0 \tag{4}$$

$$\boldsymbol{E} = \boldsymbol{E}_T = E_x\,\hat{\boldsymbol{x}} + E_y\,\hat{\boldsymbol{y}} \tag{5}$$

Transverse magnetic field $\boldsymbol{H}_t$ components are determined from Maxwell's equation $\nabla \times \boldsymbol{E} = -\mathrm{i}\omega\mu(z)\boldsymbol{H}$, and we do not need to solve the corresponding wave equation for the magnetic field. By using the variable separation approach, we could express the electric field vector $\boldsymbol{E}(x,y,z) = \boldsymbol{T}(x,y)Z(z)$, where $\boldsymbol{T}(x,y)$ and $Z(z)$ represent the transverse field vector and longitudinal field function respectively. The wave equation in (4) could now be divided into the following two equations:

$$\nabla_t^2\boldsymbol{T} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\boldsymbol{T} = 0 \tag{6}$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} - \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}z}\frac{\mathrm{d}Z}{\mathrm{d}z} + k^2(z)Z = 0 \tag{7}$$

Here we have introduced the definition of the spatially-dependent total wavenumber $k^2(z) = \omega^2\varepsilon(z)\mu(z)$, which could be explicitly expressed as follows:

$$\begin{aligned} k^2(z) &= \frac{\omega^2}{c^2}\frac{\varepsilon_{I1}+\varepsilon_{I2}}{2\beta}\frac{\mu_{I1}+\mu_{I2}}{2\beta}\left[\tanh\left(\frac{z}{z_0}\right) + \mathrm{i}\beta\right]^2 \\ &= k_0^2(\omega)\left[\tanh\left(\frac{z}{z_0}\right) + \mathrm{i}\beta\right]^2 \end{aligned} \tag{8}$$

The constant $k_0 = k_0(\omega)$ is defined as follows:

$$\begin{aligned} k_0(\omega) &= \sqrt{\frac{\omega^2}{c^2}\frac{\varepsilon_{I1}+\varepsilon_{I2}}{2\beta}\frac{\mu_{I1}+\mu_{I2}}{2\beta}} \\ &= \frac{\omega}{c}\sqrt{(\varepsilon_R - \mathrm{i}\frac{\varepsilon_{I1}-\varepsilon_{I2}}{2})(\mu_R - \mathrm{i}\frac{\mu_{I1}-\mu_{I2}}{2})} \\ &= \frac{\omega}{c}\frac{1}{2\beta}\sqrt{(\varepsilon_{I1}+\varepsilon_{I2})(\mu_{I1}+\mu_{I2})} \end{aligned} \tag{9}$$

It is here important to note that the quantity βk_0 is a real-valued spatial constant defined as follows:

$$\beta k_0 = \frac{\omega}{2c}\sqrt{(\varepsilon_{I1}+\varepsilon_{I2})(\mu_{I1}+\mu_{I2})} \tag{10}$$

Here we recall that $c = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light in free space. The importance of the quantity expressed in (10) will be explained in the subsequent section.

3 Analytical Solution For The Fields

To obtain an analytical solution for the TEM fields, we need to solve for the two differential equations (6) and (7) separately. Given that the hollow waveguide structure has an arbitrarily shaped cross-section, the differential equation (6) will be assumed solvable and equal to some vector-valued mathematical profile $\boldsymbol{T}(x,y)$. To solve the differential equation (7), we introduce the transformation $Z = \sqrt{\mu}F$ to eliminate the term with a first-order derivative over Z. This transformation yields the following result:

$$\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} + \left[\omega^2\varepsilon\mu + \frac{1}{2\mu}\frac{\mathrm{d}^2\mu}{\mathrm{d}z^2} - \frac{3}{4\mu^2}\left(\frac{\mathrm{d}\mu}{\mathrm{d}z}\right)^2\right]F = 0 \tag{11}$$

The differential equation (11) resulting from the above-introduced transformation is applicable to both lossless and lossy media. However, moving forward, we will solve the differential equation in (11) for lossless media. The case with lossy media will be treated at the end of this section. For lossless media, the permittivity and permeability functions could be expressed as follows:

$$\varepsilon(z) = -\varepsilon_0\varepsilon_r\tanh\left(\frac{z}{z_0}\right),\quad \mu(z) = -\mu_0\mu_r\tanh\left(\frac{z}{z_0}\right) \tag{12}$$

Implementing the functions in (12), and the constant $\kappa_0^2 = \varepsilon_0\mu_0\varepsilon_r\mu_r$, into equation (11) results in the following:

$$\begin{aligned} &\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} + \left[\kappa_0^2\tanh^2\left(\frac{z}{z_0}\right) - \frac{1}{z_0^2}\left(1 - \tanh^2\left(\frac{z}{z_0}\right)\right)\right. \\ &\left. - \frac{3}{4z_0^2\tanh^2\left(\frac{z}{z_0}\right)}\left(1 - \tanh^2\left(\frac{z}{z_0}\right)\right)^2\right]F = 0 \end{aligned} \tag{13}$$

Using $\mathrm{sech}^2(z/z_0) = 1 - \tanh^2(z/z_0)$ we express the result in terms of only hyperbolic tangent functions $\tanh^2(z/z_0)$. If we now introduce the new variable $u = \tanh^2(z/z_0)$ instead of z, we obtain a differential equation with respect to the variable u as follows:

$$\begin{aligned} &u(1-u)\frac{\mathrm{d}^2 F}{\mathrm{d}u^2} + \frac{1}{2}(1-3u)\frac{\mathrm{d}F}{\mathrm{d}u} \\ &-\frac{1}{4}\left[z_0^2\kappa_0^2 - \frac{z_0^2\kappa_0^2}{(1-u)} + \frac{1}{4} + \frac{3}{4u}\right]F = 0 \end{aligned} \tag{14}$$

To rewrite equation (14) such that it becomes a Gaussian hypergeometric equation, we need to eliminate the singular terms proportional to u^{-1} and $(1-u)^{-1}$. If we introduce the new function ϕ such that $\{F = u^{3/4}\phi\}$, the differential equation in (14) becomes:

$$u(1-u)\frac{\mathrm{d}^2\phi}{\mathrm{d}u^2} + (2-3u)\frac{\mathrm{d}\phi}{\mathrm{d}u} - \frac{1}{4}\left[z_0^2\kappa_0^2 - \frac{z_0^2\kappa_0^2}{(1-u)} + 4\right]\phi = 0 \tag{15}$$

To eliminate the term proportional to $(1-u)^{-1}$, we use a new function v such that $\{\phi = (1-u)^{\pm\mathrm{i}z_0\kappa_0/2}v\}$. Equation (15) then becomes:

$$u(1-u)\frac{\mathrm{d}^2 v}{\mathrm{d}u^2} + (2-(3\pm\mathrm{i}z_0\kappa_0)u)\frac{\mathrm{d}v}{\mathrm{d}u} - [1\pm\mathrm{i}z_0\kappa_0]\,v = 0 \tag{16}$$

Now that all the singular terms are eliminated, the equation (14) becomes the Gaussian hypergeometric equation. The usual form of the hypergeometric equation is:

$$u\,(1-u)\,\frac{\mathrm{d}^2 v}{\mathrm{d}u^2}+(c-(a+b+1)\,u)\,\frac{\mathrm{d}v}{\mathrm{d}u}-abv=0 \tag{17}$$

Comparing coefficients of equations (16) and (17) gives:

$$a+b=2\pm \mathrm{i}z_0\kappa_0 \quad , \quad ab=1\pm \mathrm{i}z_0\kappa_0 \tag{18}$$

Solving this equation system, we obtain the parameters

$$a=1\pm \mathrm{i}z_0\kappa_0 \quad , \quad b=1 \quad , \quad c=2 \tag{19}$$

The interchange between the coefficients a and b also solves (18) and is analogous to the coefficients in (19). Solution to the hypergeometric equation (17) is the following:

$$v(u)={}_2F_1(a,b,c;u)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{u^n}{n!} \tag{20}$$

Here ${}_2F_1(a,b,c;u)$ is the hypergeometric function. If we trace back all the substitutions in the above derivation, we obtain the following expressions for $Z(z)$:

$$\begin{aligned} Z(z) &= \sqrt{\mu}u^{3/4}\,(1-u)^{\pm \mathrm{i}z_0\kappa_0/2}\,v \\ &= \mathrm{i}\sqrt{\mu_0\mu_r}\tanh^2\left(\frac{z}{z_0}\right)\left[\cosh\left(\frac{z}{z_0}\right)\right]^{\mp \mathrm{i}z_0\kappa_0} \\ &\quad \times {}_2F_1\left[1\pm \mathrm{i}z_0\kappa_0,1,2;\tanh^2\left(\frac{z}{z_0}\right)\right] \end{aligned} \tag{21}$$

Using the following identity for hypergeometric functions [8]:

$$\tanh^2\left(\frac{z}{z_0}\right){}_2F_1\left[1\pm \mathrm{i}z_0\kappa_0,1,2;\tanh^2\left(\frac{z}{z_0}\right)\right]=1 \tag{22}$$

we obtain the simplified solution for $Z(z)$ as follows:

$$Z(z)=\mathrm{i}\sqrt{\mu_0\mu_r}\left[\cosh\left(\frac{z}{z_0}\right)\right]^{\mp \mathrm{i}z_0\kappa_0} \tag{23}$$

The electric field component $\boldsymbol{E}(\boldsymbol{r})=\boldsymbol{T}(x,y)Z(z)$ is then:

$$\boldsymbol{E}(x,y,z)=E_0\boldsymbol{T}(x,y)\left[\cosh\left(\frac{z}{z_0}\right)\right]^{\mp \mathrm{i}z_0\kappa_0} \tag{24}$$

Here E_0 is an arbitrary normalization constant. So far, the derived solutions are based on the case with lossless media. For lossy media, we obtain an analogous solution to the lossless case with an introduced attenuation constant defined as $\exp(-\beta k_0 z)$. Here we have used the radiation condition and showed that the quantity in (10) represents the exponential decay rate for the TEM waves propagating in lossy media. The solution for Z and the electric field component $\boldsymbol{E}(\boldsymbol{r})=\boldsymbol{T}(x,y)Z(z)$ in lossy media could therefore finally be expressed as follows:

$$Z(z)=\mathrm{i}\sqrt{\mu_0\mu_r}e^{-\beta k_0 z}\left[\cosh\left(\frac{z}{z_0}\right)\right]^{\mp \mathrm{i}z_0\kappa_0} \tag{25}$$

$$\boldsymbol{E}(x,y,z)=E_0\boldsymbol{T}(x,y)e^{-\beta k_0 z}\left[\cosh\left(\frac{z}{z_0}\right)\right]^{\mp \mathrm{i}z_0\kappa_0} \tag{26}$$

4 Validation of Propagation Ansatz

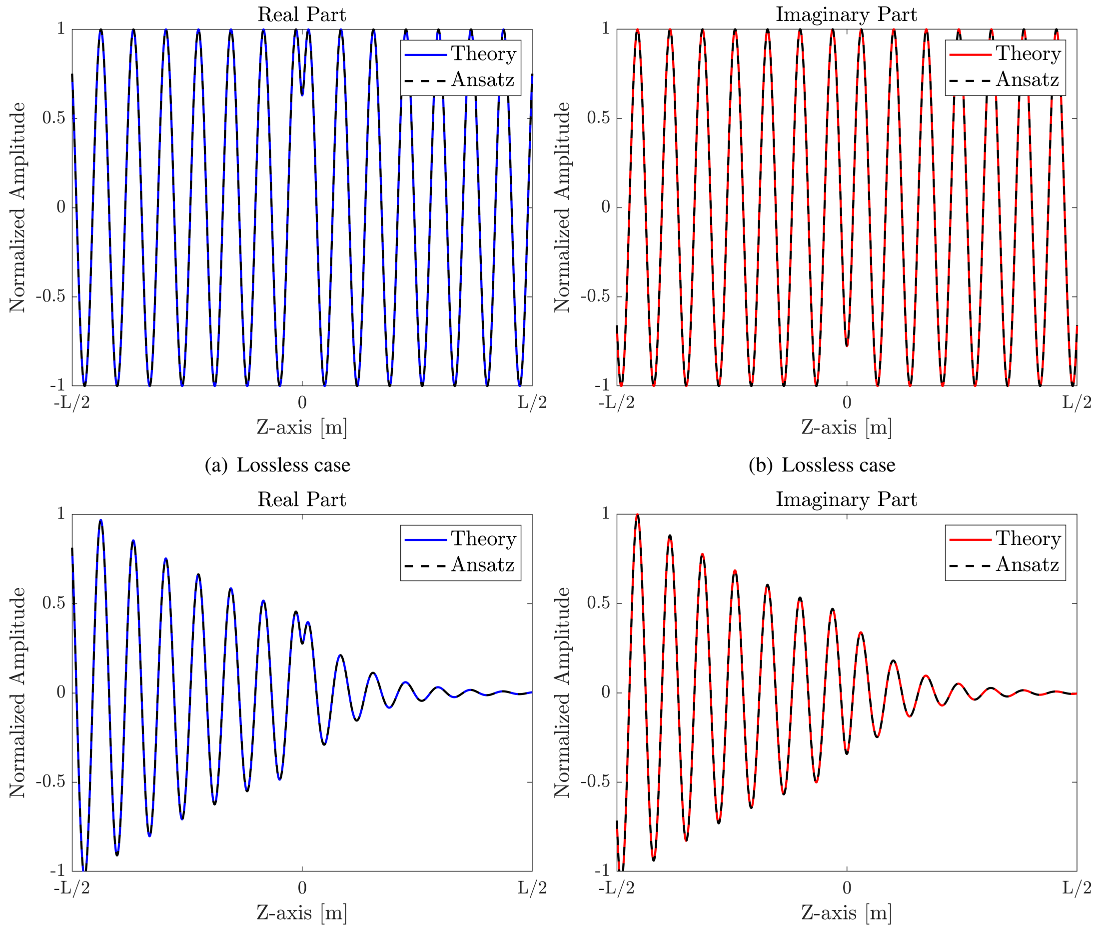
In our previous studies, the following propagation ansatz for the longitudinal field variation $Z(z)$ was used [9], [10]:

$$Z(z)=e^{-\mathrm{i}\int k(z)\mathrm{d}z} \tag{27}$$

Using MATLAB, the analytical solution in (23) and (25) was compared to the ansatz in (27) for both lossless and lossy media. In Figure 2 the real and imaginary parts of the normalized longitudinal field solution $Z(z)$ are compared for the lossless and lossy case respectively. In this example, the gradient transition of the RHM-LHM composite's permittivity and permeability functions has a steepness factor of $z_0 = 0.01$ m. The RHM has the material properties $\varepsilon = 2+0.04\mathrm{i}$ and $\mu = 1+0.02\mathrm{i}$, while the LHM has the material properties $\varepsilon = -2+0.2\mathrm{i}$ and $\mu = -1+0.1\mathrm{i}$. For this implementation, we have chosen the LHM section to be more lossy than the RHM section of our composite. This has been done in order to model a realistic RHM-LHM composite, where we know that LHMs typically have larger losses than RHMs. The operating frequency for the simulation is chosen to be 1.5 GHz and the length of the waveguide structure was chosen to be 2 m. From Figure 2, we make the same observations for the phase reversal at the interface between the RHM-LHM media and the evanescent behavior of the traveling electromagnetic wave across the waveguide structure for a lossy RHM-LHM composite as in [9]. The same observations could be made for the lossless case, except for the evanescent behavior of the traveling electromagnetic wave since the RHM-LHM composite is lossless. Furthermore, we observe that the analytical solution and the ansatz are in perfect agreement with each other, both for the lossless and lossy RHM-LHM composite. Thus, the proposed ansatz represents a unique solution to the differential equation (7), and is therefore equivalent to the analytical solution in (23) and (25) for both lossless and lossy media.

4.1 Conclusion

We investigated the longitudinal field solution for the TEM modes in waveguide structures with an arbitrarily shaped cross-section, filled with a graded impedance-matched RHM-LHM metamaterial composite. We compared the longitudinal field solution derived from the wave equation using variable separation, to a proposed ansatz of the longitudinal field solution. An excellent agreement between the analytical solution and the proposed ansatz was obtained. Thus we conclude that the proposed ansatz for the longitudinal field solution also fulfills the governing wave equation and is equivalent to the analytical solution of the longitudinal field solution. This conclusion is of interest for future work with higher-order waveguide modes where an equivalent mathematically simple ansatz can be established and verified.



(a) Lossless case (b) Lossless case

(c) Lossy case (d) Lossy case

Figure 2. Comparison between the analytical results (23) and (25) and proposed ansatz (27) for the longitudinal field solution $Z(z)$. Subfigures (a)-(b) show the results for the real and imaginary part of the normalized longitudinal field solution $Z(z)$ for lossless media, while subfigures (c)-(d) show the analogous results for lossy media.

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