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Analytic and Numerical Techniques for Computing the Fourier Spectrum of Bandlimited Periodic Signals Using Signal and Signal Derivative Sampling

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For bandlimited periodic signals, the sampling of both the signal and the signal derivative enables the number of sampling points to be reduced, and the sampling step thus to be increased, by a factor of two compared to the standard Nyquist‐Whittaker‐Shannon‐Kotelnikov‐Someya sampling theorem [1]-[2]. This alternative sampling scheme is of particular importance for spatial signals requiring a time-consuming physical movement between the sampling points. The discrete Fourier spectrum of the periodic signal can be calculated either numerically, using a system of linear equations relating the spectral coefficients to the signal and signal derivative samples, or analytically, using a closed-form relation expressing the spectral coefficients in terms of the signal and signal derivative samples [2]. This work presents a comparison between these two techniques.

A bandlimited 2π-periodic signal $w(u)$ and its first-order derivative $w'(u)$, with u being a spatial or time coordinate, can be expressed as Fourier series involving the same $2N$+1 spectral coefficients W_n,

$$w(u) = \sum_{-N}^{N} W_n e^{jnu} \quad \text{and} \quad w'(u) = \sum_{-N}^{N} jnW_n e^{jnu} .$$

In the standard approach, at least $2N$+1 signal samples at as many sampling points are required to determine the $2N$+1 spectral coefficients W_n; but using both signal and signal samples at each sampling point, it is sufficient with N+1 sampling points. From these Fourier series, a system of $2M$ x $(2N+1)$ linear equations can be established with at least $M = N + 1$ rows for the signal as well as for the signal derivative samples according to

$$\begin{aligned} w(u_m) &= (e^{-jNu_m},\ e^{-j(N-1)u_m},\ \ldots,\ 1,\ \ldots,\ e^{j(N-1)u_m},\ e^{jNu_m}) \cdot \bar{W}_n \\ w'(u_m) &= (-jNe^{-jNu_m},\ -j(N-1)e^{-j(N-1)u_m},\ \ldots,\ 0,\ \ldots,\ j(N\text{-}1)e^{j(N-1)u_m},\ jNe^{jNu_m}) \cdot \bar{W}_n \end{aligned} ,$$

where $\bar{W}_n$ is the vector of spectral coefficients; this enables a numerical solution. Alternatively, the analytic solution for the coefficients is provided by the spectrum formula [2; eq. (11)]

$$W_n = \frac{1}{M^2} \sum_{1}^{M} ((M - |n|) w(u_m) - j\frac{n}{|n|} w'(u_m))\ e^{-jnu_m} \quad \text{with} \quad u_m = u_1 + (m\text{-}1)\Delta u,\ \Delta u = \frac{2\pi}{M},\ M \geq N+1 .$$

These two techniques for computing the Fourier spectrum may differ in several aspects such as the computational complexity, the distribution of the sampling points, the sensitivity to measurement noise, and - overall - the accuracy. This presentation discusses and compares the two techniques with regard to these aspects and illustrates the findings with numerical examples based on experimental measurements of a standard gain horn antenna.

References

[1] C. E. Shannon, "Communication in the presence of noise," *Proc. IRE*, vol. 37, no. 1, pp. 10‐21, Jan. 1949.

[2] O. Breinbjerg, "Derivative Sampling of Periodic and Nonperiodic Band-Limited Signals − Fourier Spectrum and Interpolation Formula", *IEEE Trans. Antennas Propagat.*, vol. 72, no. 10, pp. 7821-7829, October 2024.