Rays and Gaussian beams related with the Feynman-Kac solutions of the Helmholtz equation

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Closed-form solutions of the Helmholtz equation $\nabla^2\phi(x)+\Omega^2K(x)\phi(x)=0$ are sought in the form

$$\phi(x)=u_+(x)\mathrm{e}^{\mathrm{i}\Omega S(x)}+u_-(x)\mathrm{e}^{-\mathrm{i}\Omega S(x)}, \qquad |\nabla S(x)|^2=\frac{K(x)}{\Omega^2}, \tag{1}$$

where $u_\pm(x)$ obey transport equations

$$\begin{aligned} \frac{\mathrm{i}}{2\Omega}\nabla^2u_+(x)-\vec{k}(x)\nabla u_+(x)-b(x)u_+(x) &=0,\\ \frac{\mathrm{i}}{2\Omega}\nabla^2u_\pm(x)+\vec{k}(x)\nabla u_-(x)+b(x)u_-(x) &=0, \end{aligned} \tag{2}$$

with $\vec{k}(x)=\nabla S(x)$ and $b(x)=\frac{1}{2}\nabla^2S(x)$. Treating $\pm\vec{k}(x)$ as two branches of the square root $\pm\sqrt{K(x)}$ on a two-sheets manifold of $\sqrt{K(z)}$, a pair of the equations (2) can be considered as a single equation on this manifold, whose solution admits the Feynman-Kac representation

$$u(z)=\left\langle u(\xi_\tau^x)\exp\left(-\frac{1}{2}\int_0^\tau\nabla b(\xi_t^x)\mathrm{d}t\right)\right\rangle, \tag{3}$$

with the averaging over all paths ξ_t described by the Ito's stochastic differential equation

$$\mathrm{d}\xi_t=-\vec{k}(\xi_t)\mathrm{d}t+\sqrt{\frac{\mathrm{i}}{\Omega}}\mathrm{d}w_t, \tag{4}$$

driven by a Brownian motion w_t. In the high frequency limit $\Omega=\infty$ the solution (3), (4) reduces to the ray method approximation

$$u(\chi)=u(\xi_\tau)\exp\left(-\frac{1}{2}\int_0^\tau\nabla b(\xi_t)\mathrm{d}t\right), \qquad \mathrm{d}\xi_t=-\vec{k}(\xi_t)\mathrm{d}t; \tag{5}$$

with the determinist trajectory described by $\mathrm{d}\xi_t=-\vec{k}(\xi_t)\mathrm{d}t$. In the case $\Omega\gg1$ the motion described by (3) and (4) may be considered as superposition of long moves $\xi_t\longrightarrow\xi_t-\vec{k}(\xi_t)\mathrm{d}t$ along rays described by a differential equation $\mathrm{d}\xi_t=-\vec{k}(\xi_t)\mathrm{d}t$, and short random moves $\xi_t\longrightarrow\xi_t\sqrt{\frac{\mathrm{i}}{\Omega}}\mathrm{d}w_t$, where $\mathrm{d}w_t$ takes random values distributed by the Gaussian law. The last statements imply that in the case $\Omega\gg1$, but $\Omega\neq\infty$, formulas (3), (4) describe Gaussian beams propagating along complex rays [2] .

References

[1] M. Freidlin. *Functional Integration and Partial Differential Equations*. Number 109 in The Annals of Mathematics Studies. Princeton University Press, Princeton, New Jersey, 1985.

[2] L. B. Felsen. *Geometrical Theory of Diffraction, Evanescent waves, Complex rays and Gaussian Beams* . Geophysical Journal of the Royal Astronomical Society, 1984, v.79, pp. 77–88.