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Well-posedness of the time-dependent, bi-anisotropic Maxwell system on anisotropic fractals

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Abstract

Anisotropic fractals are fractals which could have different, not necessarily integer, dimensions in each direction in space. We consider the time-dependent, bi-anisotropic Maxwell system on such fractals. The approach uses a homogenization method known as dimensional regularization in order to consider an equivalent continuum model in Euclidean space. Well-posedness of the Maxwell system is shown in this framework under appropriate conditions on the material parameters.

1 Introduction

A homogenization method known as dimensional regularization has been employed to obtain balance laws for fractals. This method originated in work by Tarasov [13, 14, 15], and the idea is to transform fractional integrals over fractals to equivalent, continuous problems in ambient Euclidean space in which the fractal is embedded. A product measure is used for this transformation, and the coefficients involved are related to the choice of fractional integral definition one chooses. A modification of Tarasov's framework has been developed by Li and Ostoja-Starzewski [5, 6, 8] to capture the anisotropy of a fractal; that is, the fractal solid could have different dimensions in each direction. Recently, in [11], the author has developed a basic framework to study scalar differential equations in a weak/Sobolev space setting on such fractals. This scalar framework is necessary in order to formulate the vector framework to study Maxwell's equations, or rather, the fractal variant thereof.

Here we present well-posedness of the fractal time dependent Maxwell system with general bi-anisotropic constitutive relations.

2 Background

The basic approach to homogenization of fractal media originated in the work of Tarasov [13, 14, 15]. He considers so-called fractal solids; that is, a solid having fractal geometric structure whose mass m obeys a power law with respect to the length scale (resolution)

$$m(R) = kR^D, \quad D < 3 \tag{1}$$

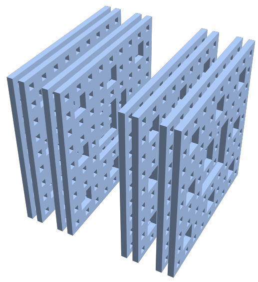

Figure 1. A few iterations of the Carpinteri column.

where D is the fractal dimension of the mass and k is a constant. In particular, Tarasov used a fractional integral to represent mass in a region $\mathscr{W}$ in $\mathbb{R}^3$. However, it could be that the medium has different Hausdorff dimensions in different directions–this is an anisotropic fractal. To account for this, another framework has been developed [5, 6, 7] which starts from a general power law relation in each coordinate:

$$m(x_1, x_2, x_3) \sim x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \tag{2}$$

The mass is then specified by a product measure:

$$m(\mathscr{W}) = \int_{\mathscr{W}} \rho(x_1, x_2, x_3) d\mu_1(x_1) d\mu_2(x_2) d\mu_3(x_3) \tag{3}$$

where the length measure $d\mu_k(x_k)$ in each coordinate is given by

$$d\mu_k(x_k) = c_1^{(k)}(\alpha_k, x_k) dx_k, \qquad k = 1,2,3 \tag{4}$$

for some coefficients $c_1^{(k)}$ which depend on the fractal dimension α_k. As has been done in [5, 8], one can take the modified Riemann-Liouville fractional integral from [3] for these coefficients:

$$c_1^{(k)}(x_k) = \alpha_k (L_k - x_k)^{\alpha_k - 1}, \qquad 0 < \alpha_k < 1, \quad k = 1,2,3 \tag{5}$$

where L_k denotes the total length of the original fractal solid along axis x_k and there is no sum on k.

As an example, consider the Carpinteri column, which has been proposed as a model composite structure with fractal type microstructures [1]. The square cross-section of this parallelepiped type domain in $\mathbb{R}^3$ is a Sierpinski carpet, which is "fractally" swept along the x_3 direction with a Cantor ternary set; see Figure 1. The Hausdorff dimension in the x_3 direction is then that of the Cantor set: $\dim_H W_3 = \ln(2)/\ln(3)$. The Hausdorff dimension of the Sierpinski carpet is known to be $\dim_H S = \ln(8)/\ln(3)$. Thus the Hausdorff dimension of this $\mathscr{W}$ is

$$\dim_H \mathscr{W} = \dim_H W_3 + \dim_H S = 4\frac{\ln(2)}{\ln(3)}$$

which is strictly smaller than 3, the dimension of the ambient Euclidean space. Now, we have the following transformation relationships between length, surface and volume elements from a fractal solid into a Euclidean parallelepiped in $\mathbb{R}^3$ with volume element dV, surface element $d\Sigma$, and length element dx:

$$\begin{aligned} d\mu_k &= c_1^{(k)}(\alpha_k, x_k)dx_k, \\ d\Sigma_d &= c_2^{(k)} d\Sigma, \\ dV_D &= c_3 dV \end{aligned} \tag{6}$$

where the volume coefficient c_3 is given by the product

$$c_3(x_1,x_2,x_3) = c_1^{(1)}(x_1)c_1^{(2)}(x_2)c_1^{(3)}(x_3)$$

The surface transformation coefficients $c_2^{(k)}$ are given by

$$c_2^{(k)} = c_1^{(i)}c_1^{(j)} = \frac{c_3}{c_1^{(k)}}, \quad i \neq j, \quad i,j \neq k$$

Again, we interpret the coefficients $dV, d\Sigma, dx$ as the appropriate elements in a homogenized continuum model of the fractal medium. For details on how these transformations are obtained, see e.g. [6].

Now, integration by parts leads (see e.g. [5]) to the notion of the fractal gradient

$$\nabla_k^D := \frac{1}{c_1^{(k)}(x_k)} \frac{\partial}{\partial x_k} \tag{7}$$

which leads to the general divergence theorem for smooth vector fields in fractal media:

$$\int_{\partial \mathscr{W}} \mathbf{f} \cdot \nu d\Sigma_d = \int_{\mathscr{W}} \left(\nabla^D \cdot \mathbf{f}\right) dV_D \tag{8}$$

That is, we can form the fractal divergence of a smooth vector field by summing the following on k:

$$\nabla^D \cdot \mathbf{f} := \nabla_k^D \cdot f_k = \frac{1}{c_1^{(k)}(x_k)} \frac{\partial f_k}{\partial x_k} \tag{9}$$

which also leads to the fractal curl of a smooth vector field

$$\left(\nabla^D \times \mathbf{f}\right)_i := \varepsilon_{ikj} \frac{1}{c_1^{(k)}(x_k)} \frac{\partial f_j}{\partial x_k} \tag{10}$$

where ε_{ikj} is the usual Levi-Civita symbol. Note that with the above notion of fractal gradient, one has that the (fractal) gradient of a constant is zero, which is not necessarily true of other fractional derivative operators. Weak versions of these operators were constructed in [11]. In particular, we see that these fractal differential operators are essentially weighted versions of the standard ones, with weights which depend on the fractal dimension and length of the fractal.

3 Fractal Maxwell equations

Assume the four fields $\mathbf{E}, \mathbf{H}, \mathbf{B}, \mathbf{D}$ and the current density $\mathbf{J}$ are smooth in space and time. The following formulation of the fractal Maxwell's equations has been obtained in [7]:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla^D \times \mathbf{E} = 0 \tag{11}$$

$$\frac{\partial \mathbf{D}}{\partial t} - \nabla^D \times \mathbf{H} = \mathbf{J} \tag{12}$$

$$\nabla^D \cdot \mathbf{B} = 0 \tag{13}$$

$$\nabla^D \cdot \mathbf{E} = 0 \tag{14}$$

In the limits that each fractal dimension $\alpha_k \to 1$, the above system reduces to the classical Maxwell system with the standard differential operators. By considering the appropriate Lagrangian, instead of (14) we supplement the other Maxwell equations with

$$\nabla^D \cdot \mathbf{D} = \rho \tag{15}$$

for a sufficiently smooth function $\rho(x,t)$.

Remark 3.1. *In [7], a variational principle is also used to arrive at an appropriate Maxwell system, which in fact is not the same system as above. Indeed, due to the anisotropic nature of $\mathscr{W}$, one can take*

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla^D (\phi c_1) \tag{16}$$

and

$$\mathbf{B} = \nabla^D \times (\mathbf{A} c_1) \tag{17}$$

in an appropriate Lagrangian, take variation with respect to ϕ and $\mathbf{A}$, respectively (these are Lagrange multipliers), then one finds that instead of (11) one has

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla^D \times \mathbf{E} = -\nabla^D \times \left(\nabla^D (\phi c_1)\right) \tag{18}$$

and instead of (13) one has

$$\nabla^D \cdot \mathbf{B} = \nabla^D \cdot \left(\nabla^D \times (\mathbf{A} c_1)\right) \tag{19}$$

The right hand sides of (18) and (19) vanish when the solid is isotropic, i.e. when $c_1^{(i)} = c_1^{(j)}$ for $i \neq j$.

4 Constitutive Relations and bi-anisotropic media

The general bi-anisotropic constitutive relations take the form

$$\begin{aligned} \mathbf{D} &= \varepsilon \mathbf{E} + \xi \mathbf{H} + \varepsilon_d * \mathbf{E} + \xi_d * \mathbf{H}, \\ \mathbf{B} &= \zeta \mathbf{E} + \mu \mathbf{H} + \zeta_d * \mathbf{E} + \mu_d * \mathbf{H} \end{aligned} \tag{20}$$

where $*$ denotes convolution in time. Each of the 8 material parameters above is a 3×3 matrix which is assumed to be essentially bounded on $\mathscr{W}$. We can define the matrices

$$A = \begin{pmatrix} \varepsilon & \xi \\ \zeta & \mu \end{pmatrix}, \qquad G(t,x) = \begin{pmatrix} \varepsilon_d & \xi_d \\ \zeta_d & \mu_d \end{pmatrix}$$

The matrix (multiplication operator) A is called the optical response and the kernel G is called the susceptibility kernel [4], and takes into account memory effects.

5 Relevant function spaces

First we need the relevant L^2 space, which in this setting is essentially just a weighted L^2 space. Indeed, define $L^2_D(\mathscr{W})$ to be the set of functions u on $\mathscr{W}$ for which $|u|^2 c_3$ is integrable. Similar to the classical function spaces, we define the fractal $\mathbf{H}(\text{curl})$ space:

$$\mathbf{H}_D(\text{f-curl}, \mathscr{W}) = \left\{ \mathbf{u} \in L^2_D(\mathscr{W})^3 : \nabla^D \times \mathbf{u} \in L^2_D(\mathscr{W})^3 \right\} \tag{21}$$

Remark 5.1. *Notice that when each $\alpha_k \to 1$, so that the total Hausdorff dimension of $\mathscr{W}$ becomes $D = 3$, then fractal curl reduces to the usual curl and the space $\boldsymbol{H}_D$ (f-curl, $\mathscr{W}$) reduces to the usual space on $\mathscr{W}$ (integrals are taken with respect to usual Lebesgue measure).*

Let us define the fractal tangential trace for a smooth vector field $\phi \in (C^\infty(\mathscr{W}))^3$:

$$\gamma_t^f(\phi) := c_2 \nu \times \phi|_{\partial\mathscr{W}} \tag{22}$$

where ν is the outer unit normal on $\partial\mathscr{W}$. Notice that the effect of the fractal is seen as a weight on the unit normal ν. The following important extension of the tangential trace is given in [12]:

Theorem 5.2. *The tangential trace map γ_t^f, originally defined on $\left(C_0^\infty(\overline{\mathscr{W}})\right)^3$, can be extended to a continuous linear map from $\boldsymbol{H}_D$ (f-curl, $\mathscr{W}$) into $\left(H^{-1/2}(\partial\mathscr{W})\right)^3$.*

Note in particular that the target space is an *unweighted* fractional space on the boundary $\partial\mathscr{W}$.

6 Well-posedness of the time dependent system

Well-posedness of the classical bi-anisotropic Maxwell system in a general setting has been studied for instance in [2], see also [10]. To setup the same problem in this homogenized fractal framework, let us consider the system (11)-(15) for $\mathbf{x} \in \mathscr{W}$ and $t > 0$. We supplement this with the following perfect conductor boundary condition: $\gamma_t^f(\mathbf{E}(\mathbf{x},t)) = 0$ on $\partial\mathscr{W}$, and initial conditions

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}), \qquad \mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}), \qquad \mathbf{x} \in \mathscr{W} \tag{23}$$

Consider the vector Hilbert space $\mathscr{H} = L^2_D(\mathscr{W})^3 \times L^2_D(\mathscr{W})^3$ and the fractal Maxwell operator $M : \mathscr{D}(M) \to \mathscr{H}$ with domain

$$\mathscr{D}(M) = \mathbf{H}_{D,0}(\text{f-curl}, \mathscr{W}) \times \mathbf{H}_D(\text{f-curl}, \mathscr{W}) \tag{24}$$

defined by

$$M = \begin{pmatrix} 0 & \nabla^D \times \\ -\nabla^D \times & 0 \end{pmatrix} \tag{25}$$

Note that the space $\mathbf{H}_{D,0}$ (f-curl, $\mathscr{W}$) has been characterized in [12], so that

$$\mathbf{H}_{D,0}(\text{f-curl}, \mathscr{W}) = \left\{ u \in \mathbf{H}_D(\text{f-curl}, \mathscr{W}) : \gamma_t^f(u) = 0 \right\}$$

just as in the classical case, i.e. when $\mathscr{W}$ is say a Lipschitz domain.

Let us define $u = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$. Then the fractal Maxwell system, with consitutive relations (20), can be written as

$$(Au + G * u)' = Mu + j \tag{26}$$

where prime denotes d/dt. Given the framework developed in [11] and [12], we can show:

Theorem 6.1. *The fractal Maxwell operator M defined in (25) with domain (24), is closed, has dense domain, and is such that $M^* = -M$. Thus, the fractal Maxwell operator $M : \boldsymbol{H}_{D,0}(\text{f-curl}, \mathscr{W}) \times \boldsymbol{H}_D(\text{f-curl}, \mathscr{W}) \to \left(L^2_D(\mathscr{W})\right)^3 \times \left(L^2_D(\mathscr{W})\right)^3$ is the generator of a unitary group $\{T_M(t)\}_{t\in\mathbb{R}}$ on $\left(L^2_D(\mathscr{W})\right)^3 \times \left(L^2_D(\mathscr{W})\right)^3$.*

Proof. By density of smooth functions in $\mathbf{H}_D$ (f-curl, $\mathscr{W}$) we have that $\mathscr{D}(M)$ is dense in $L^2_D(\mathscr{W}) \times L^2_D(\mathscr{W})$. The trace theorem in [12] further implies that M is closed. Now, we see that $(\mathbf{E},\mathbf{H}) \in \mathscr{D}(M^*)$ if and only if the map

$$(\mathbf{E},\mathbf{H}) \mapsto (\nabla^D \times \mathbf{H}, u)_{L^2_D} - (\nabla^D \times \mathbf{E}, v)_{L^2_D} \tag{27}$$

is continuous in the L^2_D topology. Note that (27) implies that $\nabla^D \times u, \nabla^D \times v \in L^2_D$, and again by the trace theorem in [12], we see that $\gamma_t^f(u) = 0$. Thus we have that $\mathscr{D}(M^*) \subseteq \mathscr{D}(M)$. Conversely, if $(u,v) \in \mathscr{D}(M)$, then for all $(\mathbf{E},\mathbf{H}) \in \mathscr{D}(M)$, we see that

$$(M(\mathbf{E},\mathbf{H}),(u,v))_{L^2_D} = ((\mathbf{E},\mathbf{H}), M(u,v))_{L^2_D}$$

by density of smooth functions in $\mathbf{H}_{D,0}$ (f-curl, $\mathscr{W}$) and $\mathbf{H}_D$ (f-curl, $\mathscr{W}$). Thus we conclude that $\mathscr{D}(M^*) = \mathscr{D}(M)$. The last part of the theorem follows by Stone's Theorem. □

Now we return to the main system (26). Assuming that the matrix G is weakly differentiable in time, we are thus led to the following problem:

$$\begin{aligned} &u'(t) = A^{-1}Mu + (-A)^{-1}G' * u + A^{-1}j, \\ &u(x,0) = u_0, \\ &\gamma_t^f u_1(x,t) = 0, \qquad x \in \partial\mathscr{W} \end{aligned} \tag{28}$$

Classical ideas can be used to ensure solvability of (28), see in particular Chapter 7 in [9]. By a classical solution of (28), we mean a function $u \in C^1([0,T]; \mathscr{H}) \cap C\left([0,T]; \mathscr{D}(A^{-1}M)\right)$ that satisfies the first equation in (28) as well as the initial condition $u(x,0) = u_0$ for all $t \in [0,T]$ for some $T > 0$. Then we have (compare with Theorem 7.4.11 in [9]):

Theorem 6.2. *Suppose that* $u_0 \in X_m := \boldsymbol{H}_{D,0}\,(f\text{-}curl, \mathscr{W}) \times \boldsymbol{H}_D\,(f\text{-}curl, \mathscr{W})$*. Suppose further that the material parameters satisfy:*

1. *There exists a constant* $c_1 > 0$ *such that* $||(-A)^{-1}G(t)y||_{\mathscr{H}} < c_1||y||_{\mathscr{H}}$ *for all* $t \in [0,T]$ *and* $y \in \mathscr{H}$*.*

2. *There exists a constant* $c_2 > 0$ *such that* $||(-A)^{-1}G(t)y||_{X_m} < c_2||y||_{X_m}$ *for all* $t \in [0,T]$ *and* $y \in X_m$*.*

3. $A^{-1}j \in L^1([0,T];X_m) \cap C([0,T];\mathscr{H})$*.*

Then there exists a unique classical solution of (28).

This follows by using a fixed point argument in the Banach space $C([0,T];X_m)$.

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