Rigorous Formulation of Vector Wave Propagation Over 3D Irregular Terrain

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Abstract – Starting from a covariant form of Maxwell's equations, a systematic way to transform the boundary value problem of electromagnetic waves propagating over a 3D terrain is presented. For the terrain-flattening transformation considered here, the uneven terrain is shown to manifest as a spatially varying, anisotropic medium with the relative permeability tensor equal to the relative permittivity tensor. Depolarization of the field is automatically incorporated into the formulation to enable a full-wave analysis of fields.

1. Introduction

Propagation of radio waves over irregular terrain is an important topic in a number of areas, including [1] radar, wireless communications, and synthetic aperture radar imaging. When waves propagate over uneven terrain, they can be reflected, scattered, and attenuated, and the amount of signal degradation or distortion depends on the undulations and composition of the terrain, as well as on the transmitting and receiving conditions. Often, one has to deal with shadowing, particularly when the transmitting and receiving points lack a line of sight. Among the mathematical techniques commonly used to study propagation over terrain are ray-tracing methods [2–4], frequency and time-domain finite-difference methods [5], and frequency-domain Fourier-transform methods [6].

Due to the electrically large nature of the terrain and the computational demands, many of the previous low-frequency techniques have focused on the 2D case, wherein the excitation and terrain geometry are assumed invariant with respect to one spatial coordinate. In this case, the vector electromagnetic problem can be broken down into two scalar problems that can be separately studied. However, a 2D approach completely obscures the aspect of depolarization, whereby a vertically polarized source generates both a vertically polarized and a horizontally polarized field and vice versa when it is placed over a general terrain. Accounting for depolarization could be particularly important in radio planning, navigation, and imaging applications, when one component of the field is completely unavailable.

When the terrain has strong variations in the meridian plane connecting the transmitter and receiver and in the lateral plane, the 2D formulations tend to be inaccurate. Even the scarce formulations currently available for the 3D terrain are only for the scalar problem that ignores depolarization [7] or are valid for unidirectional propagation coupled with nonrigorous terrain boundary implementations [8]. There is a need for a rigorous 3D formulation of vector electromagnetic waves propagating over terrain [9], which we undertake here using a covariant formulation of Maxwell's equations [10, 11].

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2. Terrain-Transformed Fields and Constitutive Parameters

Figure 1 shows an irregular terrain defined by the profile $z=\zeta(x,y)$ bounding the region of interest $-\infty<x,y<\infty$, $\zeta(x,y)<z\leq H$. For simplicity, the terrain surface S is assumed to be perfectly conducting with the electric field $\mathscr{E}$ satisfying the boundary condition $\hat{\mathbf{n}}\times\mathscr{E}=0$, where $\hat{\mathbf{n}}$ is the unit normal pointing into the region of interest. We use a transformation here, such that the uneven terrain surface gradually degenerates to a flat surface at $z=H$, as indicated in the figure. This is in contrast to the Beilis–Tappert transformation [12] $z'=z-\zeta(x,y)$, where the effect of the terrain is equally felt at all altitudes. In tensor notation, the desired forward differentials $dx^{m'}=A^{m'}_{m}dx^{m}$ and the inverse differentials $dx^{m}=A^{m}_{m'}dx^{m'}$ connecting the original coordinates $(x^1,x^2,x^3)=(x,y,z)$ and the transformed coordinates $(x^{1'},x^{2'},x^{3'})=(x',y',z')$ are

$$x^{1'}=x^1 \Rightarrow x^1=x^{1'} \tag{1}$$

$$x^{2'}=x^2 \Rightarrow x^2=x^{2'} \tag{2}$$

$$x^{3'}=(x^3-\zeta)h \Rightarrow x^3=\zeta+h^{-1}x^{3'} \tag{3}$$

where

$$h(x,y)=\frac{H}{H-\zeta(x,y)}=\frac{H}{H-\zeta(x',y')}=h(x',y') \tag{4}$$

The transformation maps the surface $z=\zeta(x,y)$ to the flat plane $z'=0$ and the plane $z=H$ to the plane $z'=H$ and is known as the σ *transformation* [13]. A 2D version of it was previously used in [14] for the twofold computational advantages it offers. As a consequence of this 3D transformation, the uneven terrain manifests here as a geometry-induced permittivity (relative) tensor $\varepsilon^{m'n'}$ and a fourth-rank constitutive (relative) tensor $C^{m'n'p'q'}$ containing the relative permeability, as shown in the following.

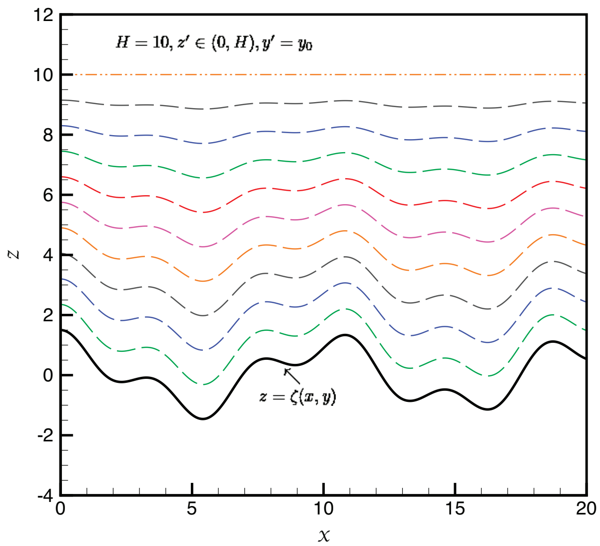

Figure 1. Inverse mapping of the lines $z'=$ constant for $z' \in (0,H)$, $y'=y_0$. The transformation flattens at $z=H$ and provides a smooth transition to $z>H$.

Using the notation $\partial_x = \partial/\partial x$, $\zeta_x = \partial_x \zeta = \zeta_{1'} = \zeta_{x'}$, $\zeta_y = \partial_y \zeta = \zeta_{2'} = \zeta_{y'}$, the matrix representations of the transformation tensors $A_m^{m'} = \partial x^{m'}/\partial x^m =: \partial_m x^{m'}$ and $A_{m'}^m = \partial x^m/\partial x^{m'} =: \partial_{m'} x^m$ are

$$[A_m^{m'}] = \begin{array}{c} \\ m'=1 \\ m'=2 \\ m'=3 \end{array} \begin{array}{c} \begin{array}{ccc} m=1 & m=2 & m=3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -gh\zeta_x & -gh\zeta_y & h \end{bmatrix} \end{array} \tag{5}$$

$$[A_{m'}^m] = \begin{array}{c} \\ m=1 \\ m=2 \\ m=3 \end{array} \begin{array}{c} \begin{array}{ccc} m'=1 & m'=2 & m'=3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g\zeta_x & g\zeta_y & h^{-1} \end{bmatrix} \end{array} = [A_m^{m'}]^{-1} \tag{6}$$

where $0 \le g(z') = \frac{H-z'}{H} \le 1$. The determinant of the Jacobian matrix is $J' = \det[A_m^{m'}] = h$, and that of the inverse transformation is $J = h^{-1}$, with both being independent of z, but dependent on (x,y). The transformation is nonorthogonal in the sense that $|J'| \neq 1$. The unit vectors in the transformed domain obey $\hat{\mathbf{x}}' \cdot \hat{\mathbf{y}}' = 0$, $\hat{\mathbf{x}}' \cdot \hat{\mathbf{z}}' = -g\zeta_x/\sqrt{1+g^2\zeta_x^2+g^2\zeta_y^2}$, and $\hat{\mathbf{y}}' \cdot \hat{\mathbf{z}}' = -g\zeta_y/\sqrt{1+g^2\zeta_x^2+g^2\zeta_y^2}$, and the last two of which are nonzero, unless $z'=H$ or ζ is a constant.

The equations of interest are the Gauss' law $\nabla \cdot \varepsilon_r \mathscr{E} = \rho_v/\varepsilon_0$ and its tensor counterpart [11, ch. 18]

$$\partial_\ell (\varepsilon^{\ell m} \mathscr{E}_m) = \frac{\rho_v}{\varepsilon_0} = \eta_0 \mathfrak{J}^4, \quad 1 \le m \le 3 \tag{7}$$

where ρ_v is the charge density that defines the *time component* $\mathfrak{J}^4 = c\rho_v$ of the four-current tensor density

Table 1. Matrix form of constitutive tensor C^{mnpq}

	C^{mnpq}		$pq \rightarrow$ 41 $\mathscr{E}_1$	42 $\mathscr{E}_2$	43 $\mathscr{E}_3$	32 $c\widetilde{\mathscr{B}}^1$	13 $c\widetilde{\mathscr{B}}^2$	21 $c\widetilde{\mathscr{B}}^3$
	41	$-c\mathscr{D}^1$	$-\varepsilon_{11}$	$-\varepsilon_{12}$	$-\varepsilon_{13}$	γ_{11}	γ_{12}	γ_{13}
mn	42	$-c\mathscr{D}^2$	$-\varepsilon_{12}$	$-\varepsilon_{22}$	$-\varepsilon_{23}$	γ_{21}	γ_{22}	γ_{23}
$\downarrow$	43	$-c\mathscr{D}^3$	$-\varepsilon_{13}$	$-\varepsilon_{23}$	$-\varepsilon_{33}$	γ_{31}	γ_{32}	γ_{33}
	32	$\widetilde{\mathscr{H}}_1$	γ_{11}	γ_{21}	γ_{31}	μ_{11}^{-1}	μ_{12}^{-1}	μ_{13}^{-1}
	13	$\widetilde{\mathscr{H}}_2$	γ_{12}	γ_{22}	γ_{32}	μ_{12}^{-1}	μ_{22}^{-1}	μ_{23}^{-1}
	21	$\widetilde{\mathscr{H}}_3$	γ_{13}	γ_{23}	γ_{33}	μ_{13}^{-1}	μ_{23}^{-1}	μ_{33}^{-1}

$\mathfrak{J}^\mu = [\mathfrak{J}^1, \mathfrak{J}^2, \mathfrak{J}^3, \mathfrak{J}^4]$, the vector wave equation $\nabla \times \mu_r^{-1} \nabla \times \mathscr{E} = -c^{-2}\varepsilon_r \ddot{\mathscr{E}} - \mu_0 \dot{\mathscr{J}}$, and its tensor counterpart [11]

$$\partial_n (C^{mnpq} \partial_p \mathscr{E}_q) = -\frac{1}{c^2} \varepsilon^{m\ell} \ddot{\mathscr{E}}_\ell - \mu_0 \ddot{\mathfrak{J}}^m, \quad 1 \le m \le 3 \tag{8}$$

An overdot here represents time derivative, and c is the speed of light in vacuum. If the same index appears in both the superscript and the subscript of a tensor or a partial derivative, a summation with respect to that index over all possible values is implied. For example, on the left-hand side (LHS) of (7), a double summation with respect to ℓ and m is implied. The constitutive tensor C^{mnpq} encompasses the permittivity ε^{mn}, the inverse permeability μ_{mn}^{-1}, and the magnetodielectric tensors γ_{mn} whose matrix form is shown in Table 1 [10, 11, pp. 18-50]. The magnetodielectric part of C^{mnpq} accounts for the cross-coupling that can occur between the electric and magnetic fields in a most general bianisotropic medium. Note that these two matrix subblocks are transposes of each other, as expected for a lossless medium [10]. The permittivity tensor satisfies the symmetry condition $\varepsilon^{mn} = \varepsilon^{nm}$, and the constitutive tensor satisfies the properties

$$C^{mnpq} = -C^{nmpq} = -C^{mnqp}; \; C^{mnpq} = C^{pqmn} \tag{9}$$

together with the alternation condition $C^{[mnpq]} = 0$ that render the number of independent unknowns in it to be 20 [11]. Because the medium above the uneven terrain is a vacuum, the permittivity tensor ε^{mn} and constitutive tensor C^{mnpq} in the original domain are the diagonal tensors

$$\varepsilon^{mn} = \begin{array}{c} \\ m=1 \\ m=2 \\ m=3 \end{array} \begin{array}{c} \begin{array}{ccc} n=1 & n=2 & n=3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \tag{10}$$

$$C^{mnpq} = \begin{array}{c} \\ mn=32 \\ mn=13 \\ mn=21 \end{array} \begin{array}{c} \begin{array}{ccc} pq=32 & pq=13 & pq=21 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \tag{11}$$

and the magnetodielectric part of C^{mnpq}, being identically zero in vacuum, will continue to be zero due to the transformation properties of C^{mnpq}.

The electric field tensor transforms as $\mathscr{E}_{m'} = A^{m}_{m'}\mathscr{E}_m$, which translates to

$$\mathscr{E}_{1'} = \mathscr{E}_1 + g\zeta_x\mathscr{E}_3,\ \mathscr{E}_{2'} = \mathscr{E}_2 + g\zeta_y\mathscr{E}_3,\ \mathscr{E}_{3'} = h^{-1}\mathscr{E}_3 \tag{12}$$

The permittivity and constitutive tensors transform as densities of weight +1 [11]

$$\varepsilon^{\ell' m'} = JA^{\ell'}_{\ell}A^{m'}_{m}\varepsilon^{\ell m};\ C^{m'n'p'q'} = JA^{m'}_{m}A^{n'}_{n}A^{p'}_{p}A^{q'}_{q}C^{mnpq} \tag{13}$$

The boundary condition on the terrain $(\hat{\mathbf{z}} - \hat{\mathbf{x}}\zeta_x - \hat{\mathbf{y}}\zeta_y)\times(\hat{\mathbf{x}}\mathscr{E}_1 + \hat{\mathbf{y}}\mathscr{E}_2 + \hat{\mathbf{z}}\mathscr{E}_3) = 0$ is transformed to

$$\mathscr{E}_{1'} = 0 = \mathscr{E}_{2'} \text{ on } S' \tag{14}$$

because $g(z'=0)=1$. Substituting the transformation matrices (5) and (6) into (13), the transformed relative permittivity tensor $\boldsymbol{\varepsilon} = \{\varepsilon^{m'n'}\}$ and the relative inverse permeability tensor $\boldsymbol{\mu}^{-1}$, that is, the part of $\{C^{m'n'p'q'}\}$ with indices indicated in (11), are

$$\varepsilon^{m'n'} = \begin{bmatrix} h^{-1} & 0 & -g\zeta_x \\ 0 & h^{-1} & -g\zeta_y \\ -g\zeta_x & -g\zeta_y & h[g^2(\zeta_x^2+\zeta_y^2)+1] \end{bmatrix} \tag{15}$$

and

$$C^{m'n'p'q'} = \begin{bmatrix} h(1+g^2\zeta_x^2) & hg^2\zeta_x\zeta_y & g\zeta_x \\ hg^2\zeta_x\zeta_y & h(1+g^2\zeta_y^2) & g\zeta_y \\ g\zeta_x & g\zeta_y & h^{-1} \end{bmatrix} \tag{16}$$

For instance, the top left entry of $C^{m'n'p'q'}$ is associated with $m'n'p'q' = 3'2'3'2'$. The properties listed in (9) determine the remaining values of the tensor $C^{m'n'p'q'}$ not shown in (16). It is relatively straightforward to obtain these transformed equations. For example, the transformed permittivity component

$$\varepsilon^{3'3'} = JA^{3'}_{m}A^{3'}_{n}\varepsilon^{mn} = h(g^2\zeta_x^2 + g^2\zeta_y^2 + 1) \tag{17}$$

and the transformed constitutive component

$$\begin{aligned} C^{1'3'3'2'} &= JA^{1'}_{m}A^{3'}_{n}A^{3'}_{p}A^{2'}_{q}C^{mnpq} = h^{-1}A^{3'}_{n}A^{3'}_{p}C^{1np2} \\ &= h^{-1}A^{3'}_{2}A^{3'}_{1}C^{1212} = hg^2\zeta_x\zeta_y \end{aligned} \tag{18}$$

Note that $\boldsymbol{\varepsilon}\boldsymbol{\mu}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix, so $\boldsymbol{\varepsilon} = \boldsymbol{\mu}$. In other words, under some very general settings, the irregular terrain induces a relative permittivity tensor identical to the relative permeability tensor in the transformed domain. Moreover, the permittivity and permeability tensors are, in general, inhomogeneous and anisotropic.

2.1 Transformed Differential Equations

Using the transformed permittivity given in (15) and the transformed derivative operators given by the chain rule $\partial_{m'} = A^{m}_{m'}\partial_m$, Gauss' law in a source-free region in the transformed domain becomes

$$\partial_{\ell'}(\varepsilon^{\ell' m'}\mathscr{E}_{m'}) = \mathscr{P}_{1'}\mathscr{E}_{1'} + \mathscr{P}_{2'}\mathscr{E}_{2'} + \mathscr{P}_{3'}\mathscr{E}_{3'} = 0 \tag{19}$$

where the operators $\mathscr{P}_{m'}(\cdot)$ are

$$\begin{aligned} \mathscr{P}_{1'} &= \partial_{1'}h^{-1} - \zeta_{1'}\partial_{3'}g = h^{-1}\partial_{1'} - \zeta_{1'}g\partial_{3'} \\ &\to \partial_{1'} \text{ as } \zeta \to 0 \end{aligned} \tag{20}$$

$$\begin{aligned} \mathscr{P}_{2'} &= \partial_{2'}h^{-1} - \zeta_{2'}\partial_{3'}g = h^{-1}\partial_{2'} - \zeta_{2'}g\partial_{3'} \\ &\to \partial_{2'} \text{ as } \zeta \to 0 \end{aligned} \tag{21}$$

$$\begin{aligned} \mathscr{P}_{3'} &= h\partial_{3'} - hg(\zeta_{1'}\mathscr{P}_{1'} + \zeta_{2'}\mathscr{P}_{2'}) \\ &\quad -g\left[\zeta_{1'1'} + \zeta_{2'2'} + \frac{2h}{H}(\zeta_{1'}^2 + \zeta_{2'}^2)\right] \\ &\to \partial_{3'} \text{ as } \zeta \to 0 \end{aligned} \tag{22}$$

In (22), the quantity $\zeta_{m'n'} := \partial^2_{m'n'}\zeta,\ m', n' = 1, 2$. On a flat terrain, $\zeta = 0$, $h = 1$, and $\mathscr{P}_{1'} = \partial_{1'}, \mathscr{P}_{2'} = \partial_{2'}$, $\mathscr{P}_{3'} = \partial_{3'}$, and one recovers the ordinary solenoidal equation $\partial_{1'}\mathscr{E}_{1'} + \partial_{2'}\mathscr{E}_{2'} + \partial_{3'}\mathscr{E}_{3'} = 0$ for the electric field.

In the transformed domain, we can represent the tensor $-\varepsilon^{m'\ell'}/c^2$ appearing on the right-hand side (RHS) of the wave equations (see 8) in a source-free region by an ordinary matrix $\mathscr{R} = \{\mathscr{R}_{m'\ell'}\}, m', \ell' = 1', 2', 3'$. The RHSs corresponding to the three rows of the wave equations are

$$m' = 1' : \sum_{\ell'=1}^{3'}\mathscr{R}_{1'\ell'}\ddot{\mathscr{E}}_{\ell'} = -\frac{1}{c^2}(h^{-1}\ddot{\mathscr{E}}_{1'} - g\zeta_{1'}\ddot{\mathscr{E}}_{3'}) \tag{23}$$

$$m' = 2' : \sum_{\ell'=1'}^{3'}\mathscr{R}_{2'\ell'}\ddot{\mathscr{E}}_{\ell'} = -\frac{1}{c^2}(h^{-1}\ddot{\mathscr{E}}_{2'} - g\zeta_{2'}\ddot{\mathscr{E}}_{3'}) \tag{24}$$

$$\begin{aligned} m' = 3' : \sum_{\ell'=1'}^{3'}\mathscr{R}_{3'\ell'}\ddot{\mathscr{E}}_{\ell'} = \frac{1}{c^2}\Big(&g\zeta_{1'}\ddot{\mathscr{E}}_{1'} + g\zeta_{2'}\ddot{\mathscr{E}}_{2'} \\ &- h[1 + g^2(\zeta_{1'}^2 + \zeta_{2'}^2)]\ddot{\mathscr{E}}_{3'}\Big) \end{aligned} \tag{25}$$

The matrix $\mathscr{R}$ is symmetric in that $\mathscr{R}_{m'\ell'} = \mathscr{R}_{\ell'm'}$. In the limit as $\zeta \to 0$, the matrix $\mathscr{R} \to -\mathbf{I}/c^2$.

The LHS of the wave equations in the transformed domain corresponding to the various indices can be expressed as

$$m' = 1' : \mathscr{L}_{1'1'}\mathscr{E}_{1'} + \mathscr{L}_{1'2'}\mathscr{E}_{2'} + \mathscr{L}_{1'3'}\mathscr{E}_{3'} \tag{26}$$

$$m' = 2' : \mathscr{L}_{2'1'}\mathscr{E}_{1'} + \mathscr{L}_{2'2'}\mathscr{E}_{2'} + \mathscr{L}_{2'3'}\mathscr{E}_{3'} \tag{27}$$

$$m' = 3' : \mathscr{L}_{3'1'}\mathscr{E}_{1'} + \mathscr{L}_{3'2'}\mathscr{E}_{2'} + \mathscr{L}_{3'3'}\mathscr{E}_{3'} \tag{28}$$

where the operators $\mathscr{L}_{m'n'}(\cdot)$ can be shown to be

$$\mathscr{L}_{1'1'} = -\mathscr{P}_{2'}h\mathscr{P}_{2'} - h\partial_{3'}^2 \tag{29}$$

$$\mathscr{L}_{1'2'} = \mathscr{P}_{1'}h\mathscr{P}_{2'} - \frac{\zeta_{1'2'}}{H} \tag{30}$$

$$\begin{aligned}\mathscr{L}_{1'3'} &= \partial_{1'}\mathscr{P}_{3'} + g(\partial_{1'}^2 + \partial_{2'}^2)\zeta_{1'} - (\partial_{1'}h\zeta_{1'}^2 + \partial_{2'}h\zeta_{1'}\zeta_{2'})\\ &\quad \partial_{3'}g^2 - \frac{h^2\zeta_{1'}}{H}\partial_{3'} - h(\zeta_{2'}\zeta_{2'1'} - \zeta_{1'}\zeta_{2'2'})\partial_{3'}g^2\end{aligned} \tag{31}$$

$$\mathscr{L}_{2'1'} = \mathscr{P}_{2'}h\mathscr{P}_{1'} - \frac{\zeta_{2'1'}}{H} \tag{32}$$

$$\mathscr{L}_{2'2'} = -\mathscr{P}_{1'}h\mathscr{P}_{1'} - h\partial_{3'}^2 \tag{33}$$

$$\begin{aligned}\mathscr{L}_{2'3'} &= \partial_{2'}\mathscr{P}_{3'} + g(\partial_{2'}^2 + \partial_{1'}^2)\zeta_{2'} - (\partial_{2'}h\zeta_{2'}^2 + \partial_{1'}h\zeta_{2'}\zeta_{1'})\\ &\quad \partial_{3'}g^2 - \frac{h^2\zeta_{2'}}{H}\partial_{3'} - h(\zeta_{1'}\zeta_{1'2'} - \zeta_{2'}\zeta_{1'1'})\partial_{3'}g^2\end{aligned} \tag{34}$$

$$\begin{aligned}\mathscr{L}_{3'1'} &= h(\partial_{3'} + g\zeta_{2'2'})\mathscr{P}_{1'} + g\partial_{2'}h(\zeta_{1'}\mathscr{P}_{2'} - \zeta_{2'}\mathscr{P}_{1'})\\ &\quad +\zeta_{1'}gh^2\partial_{3'}^2 + gh(\zeta_{2'2'}\mathscr{P}_{1'} - \zeta_{1'2'}\mathscr{P}_{2'})\end{aligned} \tag{35}$$

$$\begin{aligned}\mathscr{L}_{3'2'} &= h(\partial_{3'} + g\zeta_{1'1'})\mathscr{P}_{2'} + g\partial_{1'}h(\zeta_{2'}\mathscr{P}_{1'} - \zeta_{1'}\mathscr{P}_{2'})\\ &\quad +\zeta_{2'}gh^2\partial_{3'}^2 + gh(\zeta_{1'1'}\mathscr{P}_{2'} - \zeta_{2'1'}\mathscr{P}_{1'})\end{aligned} \tag{36}$$

$$\mathscr{L}_{3'3'} = -(\partial_{1'}h\partial_{1'} + \partial_{2'}h\partial_{2'}) - h^3g^2(\zeta_{1'}\mathscr{P}_{2'} - \zeta_{2'}\mathscr{P}_{1'})^2 \tag{37}$$

There is some regularity in the operators (29) to (37) with respect to the indices. They satisfy the interchange properties that $\mathscr{L}_{j'i'} = \mathscr{L}_{i'j'}|_{1'\rightleftarrows 2'}$, $\mathscr{L}_{i'3'} = \mathscr{L}_{j'3'}|_{1'\rightleftarrows 2'}$, and $\mathscr{L}_{3'i'} = \mathscr{L}_{3'j'}|_{1'\rightleftarrows 2'}$, $1' \leq (i', j') \leq 2'$.

To obtain differential equations for the electric field, consider first the equation pertaining to the index $m' = 1'$ in the wave equations. We use the operator identity $\partial_{1'}\mathscr{P}_{3'} = \mathscr{P}_{1'}h\mathscr{P}_{3'} + \zeta_{1'}h\partial_{3'}g\mathscr{P}_{3'}$ and consider $\mathscr{L}_{1'1'}\mathscr{E}_{1'} + \mathscr{L}_{1'2'}\mathscr{E}_{2'} + \mathscr{L}_{1'3'}\mathscr{E}_{3'}$ together with the RHS (23) and Gauss' law (19). It can be shown that

$$\begin{aligned}&(\mathscr{P}_{1'}h\mathscr{P}_{1'} + \mathscr{P}_{2'}h\mathscr{P}_{2'} + h\partial_{3'}^2)\mathscr{E}_{1'} - \zeta_{1'}h\partial_{3'}g\mathscr{P}_{3'}\mathscr{E}_{3'}\\ &\quad -g(\partial_{1'}^2 + \partial_{2'}^2)\zeta_{1'}\mathscr{E}_{3'} + (\partial_{1'}h\zeta_{1'}^2 + \partial_{2'}h\zeta_{1'}\zeta_{2'})\partial_{3'}g^2\mathscr{E}_{3'}\\ &\quad +h(\zeta_{2'}\zeta_{2'1'} - \zeta_{1'}\zeta_{2'2'})\partial_{3'}g^2\mathscr{E}_{3'} + \frac{h^2\zeta_{1'}}{H}\partial_{3'}\mathscr{E}_{3'}\\ &\quad = -\sum_{n'=1'}^{3'}\mathscr{R}_{1'n'}\ddot{\mathscr{E}}_{n'}\end{aligned} \tag{38}$$

A second differential equation in the field variable $\mathscr{E}_{2'}$ is obtained from (38) by the interchange $1' \rightleftarrows 2'$. The third differential equation is obtained by considering the expression $\mathscr{L}_{3'1'}\mathscr{E}_{1'} + L_{3'2'}\mathscr{E}_{2'} + \mathscr{L}_{3'3'}\mathscr{E}_{3'}$ together with the RHS from (25). The simplified equation reads

$$\begin{aligned}&(\partial_{1'}h\partial_{1'} + \partial_{2'}h\partial_{2'} + h\partial_{3'}\mathscr{P}_{3'})\mathscr{E}_{3'} - \zeta_{1'}gh^2\partial_{3'}^2\mathscr{E}_{1'}\\ &\quad -\zeta_{2'}gh^2\partial_{3'}^2\mathscr{E}_{2'} - g\partial_{2'}h(\zeta_{1'}\mathscr{P}_{2'} - \zeta_{2'}\mathscr{P}_{1'})\mathscr{E}_{1'}\\ &\quad - g\partial_{1'}h(\zeta_{2'}\mathscr{P}_{1'} - \zeta_{1'}\mathscr{P}_{2'})\mathscr{E}_{2'}\\ &\quad - gh(2\zeta_{2'2'}\mathscr{P}_{1'} - \zeta_{1'2'}\mathscr{P}_{2'})\mathscr{E}_{1'}\\ &\quad - gh(2\zeta_{1'1'}\mathscr{P}_{2'} - \zeta_{2'1'}\mathscr{P}_{1'})\mathscr{E}_{2'}\\ &\quad - h^3g^2(\zeta_{1'}\mathscr{P}_{2'} - \zeta_{2'}\mathscr{P}_{1'})^2\mathscr{E}_{3'}\\ &\quad = -\sum_{n'=1'}^{3'}\mathscr{R}_{3'n'}\ddot{\mathscr{E}}_{n'}\end{aligned} \tag{39}$$

In the limit as $\zeta \to 0$, one recovers the usual wave equation in the field variables $\mathscr{E}_{1'}$ and $\mathscr{E}_{3'}$ from (38) and (39), respectively. Note that (39) is invariant with respect to the transformation $1' \rightleftarrows 2'$.

As an extreme special case, if the terrain is invariant in the y direction and the electric field polarized in the y direction, then $\zeta_{2'} = 0$ and $\mathscr{P}_{2'} = 0$. Thus, (38) and (39) are satisfied with $\mathscr{E}_{1'} = 0 = \mathscr{E}_{3'}$. The dual of equation (38) simplifies to the scalar uncoupled equation

$$[(\partial_{1'} - \zeta_x hg\partial_{3'})^2 + h^2\partial_{3'}^2]\mathscr{E}_{2'} = \frac{1}{c^2}\ddot{\mathscr{E}}_{2'} \tag{40}$$

where $\zeta_{1'} = \zeta_x$ has been used. There is no depolarization of fields in this special case, and the transformed electric field will continue to have only the copolar component.

2.2 Transformed Absorbing Boundary Condition

Because the coordinate line flattens at the upper end, it is very convenient to implement an absorbing boundary condition at $z = H$ with the current transformation. For instance, the first-order Higdon-type absorbing boundary condition [15] and its transformed form are

$$\sin\alpha\,\dot{\mathscr{E}}_3 + c\frac{\partial\mathscr{E}_3}{\partial z} = 0;\ \sin\alpha\,\dot{\mathscr{E}}_{3'} + ch\frac{\partial\mathscr{E}_{3'}}{\partial z'} = 0 \tag{41}$$

given that $\mathscr{E}_3 = h\mathscr{E}_{3'}$, $\partial/\partial z = h\partial/\partial z'$, and $\partial h/\partial z' = 0$. Perfect transmission through the boundary $z = H$ takes place with the Higdon boundary condition at the grazing angles $\alpha, \pi - \alpha$ with respect to the plane $z = H$.

3. Conclusions

The field equations derived here, (38) and (39), are exact for the transformation considered. It is not necessary to simultaneously solve for all three components of the electric field in view of (19). One could make suitable approximations in the final equations, depending on the order of accuracy desired. For example, in a first-order solution, all derivatives of the terrain profile beyond the first order can be ignored. Some simplifications can be made if one assumes that waves propagate in a forward direction with respect to a preferred direction. This approximation becomes valid if the slopes of the terrain are shallow in the preferred direction and if the transmitter and receiver are in the line of sight. In particular, efficient formulations over slowly varying terrains by the parabolic equation approach can be devised from the equations presented here. It is straightforward to extend the formulation to a terrain with an impedance boundary condition. In this case, the transformation properties of the second-rank magnetic tensor B_{mn} [10, p. 149] will also be involved, and this will be undertaken in the future. If the terrain is invariant with respect to one coordinate, a 2D-scalar formulation with decoupled transverse electric and transverse magnetic waves is possible as in (40). Finally, the time-harmonic case can be handled by substituting $-\ddot{\mathscr{E}}/c^2 = k^2\mathscr{E}$, where k is the wave number at frequency ω. Detailed numerical results with these formulations will be separately presented.

References

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