Electromagnetic Field Interaction With a Gainy Slab

Ari Sihvola

Abstract – This article focuses on the one-dimensional reflection–transmission problem from a dielectric slab. The material of the slab is isotropic, but the permittivity is complex, allowing the medium be dissipative, lossless, or active. In the active case, the power propagation within the slab is intricate, leading to interesting corollaries concerning poles and resonances of the structure. Furthermore, the textbook reflection coefficient formula for the half-space reflection is revisited.

1. Introduction

Although the interaction of electromagnetic waves with dielectric and magnetic materials has a long history, the response of active media (media characterized by gain) has not received very much attention. This is understandable because the realization of continuous media with gain is not a straightforward engineering task. However, technology expands its frontiers; hence, it is worthwhile to conduct theoretical studies on the interaction of waves with active media. Ways to achieve gain in the material response vary from application of pyroelectric or piezoelectric materials, incorporation of pumping lasers, or composite designs with inclusions of parametrically amplified electronic components into a neutral host matrix [1, 2]. The purpose of this article is to point out some nontrivial phenomena observed in a very simple setting of a wave hitting a gainy slab. This problem can be seen as a one-dimensional parallel to previous studies of scattering by three-dimensional active objects [3, 4].

The problem is illustrated in Figure 1: a plane wave from free space (ε_0, μ_0) is normally incident on a planar slab with thickness d. The slab is dielectric $(\mu = \mu_0)$ with complex relative permittivity $\varepsilon = \varepsilon' - \mathrm{j}\,\varepsilon''$; hence, its refractive index is $n = \sqrt{\varepsilon} = n' - \mathrm{j}\,n''$. Note the sign of the imaginary part of the permittivity: the time-harmonic function is $\exp(\mathrm{j}\omega t)$, which means that dissipative media have $\varepsilon'' > 0$, while for gainy media, the imaginary part is positive (ε'' and n'' are negative).

2. Reflection and Transmission

For normal incidence, the electric and magnetic fields in Figure 1 are in the xy plane. Denoting $k = \omega\sqrt{\mu_0\varepsilon_0}$ as the free-space wave number, the normalized fields (assuming unit incidence) read

$$E(z) = \mathrm{e}^{-\mathrm{j}kz} + R\mathrm{e}^{+\mathrm{j}kz} \quad z < 0 \tag{1}$$

$$= A\mathrm{e}^{-\mathrm{j}nkz} + B\mathrm{e}^{+\mathrm{j}nkz} \quad 0 < z < d \tag{2}$$

$$= T\mathrm{e}^{-\mathrm{j}kz} \quad d < z \tag{3}$$

The field coefficients can be solved from the interface conditions (continuity of the tangential electric and magnetic fields) at the interfaces $z = 0$ and $z = d$, according to standard electromagnetic textbook treatments [5, 6]:

$$R = \frac{(n+1)(n-1)\left(1 - \mathrm{e}^{2\mathrm{j}nkd}\right)}{\mathrm{e}^{2\mathrm{j}nkd}(n+1)^2 - (n-1)^2} \tag{4}$$

$$A = \frac{2(n+1)\mathrm{e}^{2\mathrm{j}nkd}}{\mathrm{e}^{2\mathrm{j}nkd}(n+1)^2 - (n-1)^2} \tag{5}$$

$$B = \frac{2(n-1)}{\mathrm{e}^{2\mathrm{j}nkd}(n+1)^2 - (n-1)^2} \tag{6}$$

$$T = \frac{4n\,\mathrm{e}^{\mathrm{j}nkd}}{\mathrm{e}^{2\mathrm{j}nkd}(n+1)^2 - (n-1)^2} \tag{7}$$

The real part of the Poynting vector $S = \frac{1}{2}E \times H^*$ gives the time-averaged flow of power. For a homogeneous plane wave propagating in the z direction in a dielectric medium, the complex Poynting vector reads

$$S = \frac{1}{2\eta^*}\,|E|^2\,u_z = \frac{u_z}{2\eta_0}\,n^*\,|E|^2 \tag{8}$$

where $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the free-space wave impedance. Normalizing the incident power density to unity, the real part of the normalized Poynting vector toward the $+z$ direction is $1 - |R|^2$ on the left side of the slab, and after the slab $(z > d,\ n = 1)$, it is $|T|^2$. Within the slab $(0 < z < d, n = n' - \mathrm{j}n'')$, the real part of the z component of $2\eta_0 S$ is

$$n'\left(|A|^2\,\mathrm{e}^{-2n''kz} - |B|^2\,\mathrm{e}^{+2n''kz}\right) - 2n''\,\mathrm{Im}\{AB^*\,\mathrm{e}^{-\mathrm{j}2n'kz}\} \tag{9}$$

For a lossless medium $(n'' = 0)$, this expression distills to $n(|A|^2 - |B|^2)$, but for dissipation or gain, there is an interaction term also in the real part of the Poynting

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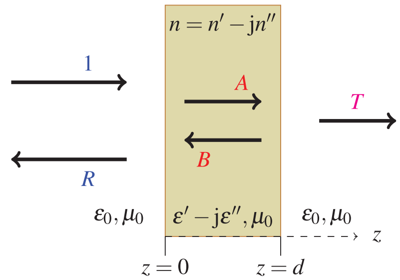

Figure 1. The geometry of the slab problem with thickness d. A plane wave is normally incident, reflected with (electric field) coefficient R, creating internal forward (A) and backward (B) propagating fields, and transmitted with coefficient T.

vector, leading to a nonsmooth behavior of the power propagation within the slab.

2.1 Slab With Passive Material

Starting with a lossless slab (real ε), Figure 2 shows the absolute value of the fields for slab thickness $kd = 3$, for two values of the refractive index n. As is known from basic electromagnetic textbooks [5, 6], there is interference for the fields in the slab and in the reflection region $z < 0$. Note that in both lossless cases, the propagating power to the right is the same in all three regions. For $n = 1.5$ (blue solid line), the values are $|R| \approx 0.377$, $|T| \approx 0.926$, satisfying $|R|^2 + |T|^2 = 1$. As seen in the figure, the field within the slab is, on average, smaller than in the air regions; however, the propagating power is the same: $1 - |R|^2 = |T|^2 = n(|A|^2 - |B|^2) \approx 0.8577$.

For a lossy slab ($\varepsilon'' > 0$), part of the energy is absorbed in the dielectric losses of the slab. Figure 3 depicts the behavior of the absolute value of the total electric field for two lossy cases. Now, the reflection and transmission values do not add up to unity, because the share of the absorbed power is $1 - |R|^2 - |T|^2$. For example, for the blue curve ($n = 1.5 - \mathrm{j}0.1$), the numbers are $|R|^2 = 0.0914$, $|T|^2 = 0.4842$, and $1 - |R|^2 - |T|^2 = 0.4244$. In Figure 4, the real part of the $+z$ component of the Poynting vector is shown. It is positive in all three regions, but in the slab, it has negative divergence.

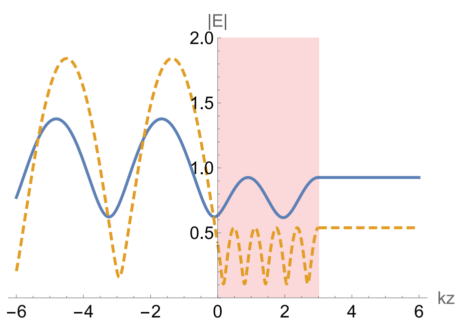

Figure 2. The absolute value of the total electric field: slab refractive index (blue solid line) $n = 1.5$; dashed line (orange) $n = 5$. The thickness of the slab is $kd = 3$, with k being the free-space wave number.

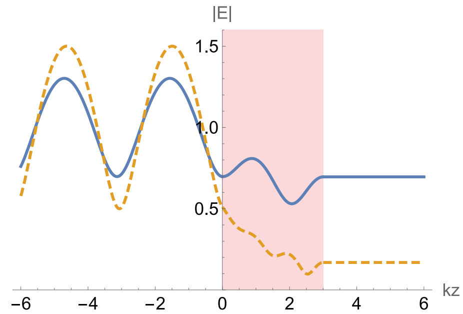

Figure 3. The same as in Figure 2, for a lossy slab with width $kd = 3$: slab refractive index (blue solid line) $n = 1.5 - \mathrm{j}\,0.1$; dashed line (orange) $n = 3 - \mathrm{j}\,0.5$.

Also, note its nonuniform decrease within the slab, in agreement with (9).

2.2 Slab With Active Dielectric Material

For an active slab ($\varepsilon'' < 0$), we can use the derived results to compute reflection and transmission but care has to be taken. Because active materials generate energy, as opposed to passive media (that are lossless and dissipative), the sum of reflected and transmitted powers exceeds that of the incident power of the incoming plane wave.

This is illustrated by two examples in Figure 5. Again, $kd = 3$, but the medium is active with $n = 2 + \mathrm{j}\,0.1$ and $n = 2 + \mathrm{j}$. The power balance for $n = 2 + \mathrm{j}0.1$ is the following: $|R|^2 = 0.191$; $|T|^2 = 1.975$; and $1 - |R|^2 - |T|^2 = -1.167$. This negative *absorbance* means that the slab generates energy flowing into the left and right.

For the smaller amount of gain ($n = 2 + \mathrm{j}\,0.1$), there is a moderate reflection, but the magnitude of the transmission coefficient T is over unity (the amplitude of the transmitted electric field is larger than the incident field). However, when the gain increases ($n = 2 + \mathrm{j}$), the transmission becomes very small, but the reflection coefficient considerably increases. Hence, even if the slab is active, there is not much transmitted energy in this case.

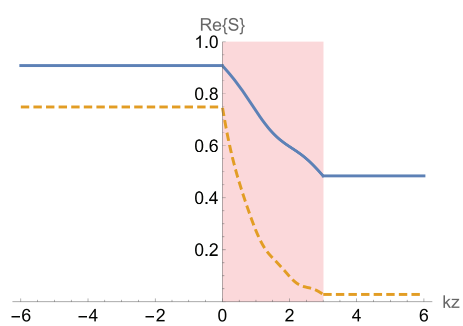

Figure 4. The real part of the $+z$ component of the normalized Poynting vector for the cases in Figure 3. Note the negative divergence of the Poynting vector (decreasing in its own direction.)

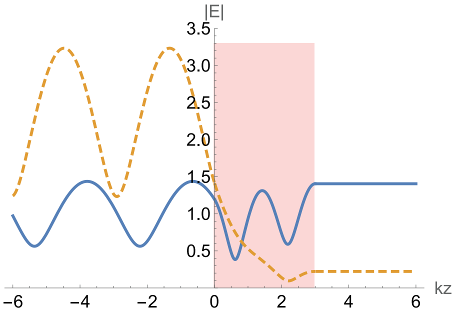

Figure 5. Total electric field (as in Figures 2 and 3) for an active slab with width $kd = 3$: slab refractive index (blue solid line) $n = 2 + \mathrm{j}\,0.1$; dashed line (orange) $n = 2 + \mathrm{j}$.

This phenomenon is clearly seen from the behavior of the Poynting vector (cf. Figure 6). The $+z$ component of the real part of the Poynting vector is plotted and normalized so that the incident power density is unity. For both cases, the $+z$ component of the Poynting vector is larger on the right-hand side of the slab than on its right side.

Here again, the real part of the Poynting vector into the z direction equals $1 - |R|^2$ on the left side of the slab and $|T|^2$ on its right side. In the low-gain case ($n = 2 + \mathrm{j}\,0.1$), the power propagates into $+z$ in all three regions. The real part of the Poynting vector is constant in the lossless regions, but not in the slab, because there, it has positive divergence.

A radical difference between the two cases is that in the second case ($n = 2 + \mathrm{j}$), the z component changes sign. It is negative for $z < 0$, which means that the net power is propagating away from the slab (the reflection coefficient is more than unity, and the reflected power into $-z$ is five times stronger than the incident power into $+z$). There is a small amount of transmitted energy, meaning that the component is positive, and due to continuity, it has to change sign within the slab. There is a point ($kz = 2.7357$) at which the real part of the Poynting vector vanishes. On its left side, the power is sent back to $-z$, while on its right side, the power is pushed into the transmission direction. Note, however, that this zero Poynting position is not the same location, when the electric field magnitude is minimum (Figure 5).

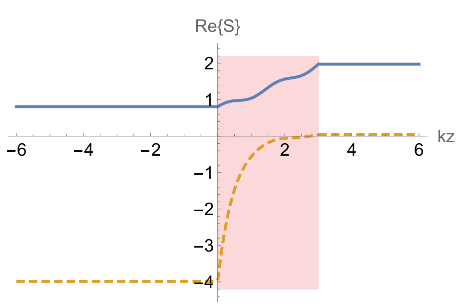

Figure 6. The real part of the $+z$ component of the normalized Poynting vector for the cases in Figure 5. Note the positive divergence of the Poynting vector within the slab (increasing in its own direction.)

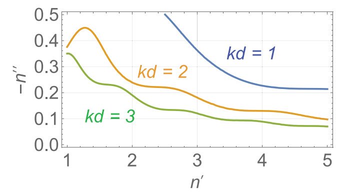

Figure 7. The conditions for an active slab with thickness d and complex refractive index n reflect the incident plane wave with unit magnitude.

3. Repercussions for Active Slab Interaction

3.1 Unity and Reflection Slab

For the reflection from an active slab, Figure 6 shows that the real part of the Poynting vector in front of the slab can be positive or negative; hence, for certain slab parameters, it could vanish. In other words, there is no net flow of power in the region $z < 0$: the amount of reflected power is the same as that of the incident wave ($|R| = 1$). Only a standing wave remains. This situation depends on the thickness of the slab and its complex refractive index. Figure 7 shows the values for this to happen in the complex n plane for three different slab thicknesses $kd = 1, 2, 3$.

3.2 Reflection Resonances From an Active Slab

For lossless slabs, the so-called *half-wavelength window* effect is known: the reflection vanishes for a frequency with the condition $nkd = \pi$. Such a situation does not take place, if $n'' \neq 0$. However, another resonance-type phenomenon can emerge in the active domain: strong resonances can appear that make all the coefficients R, T, A, and B singular. This happens when the denominator in (4)–(7) vanishes. Because it is complex, both the real and imaginary parts need to vanish for a given combination of the slab thickness d and complex refractive index $n = n' - \mathrm{j}\,n''$ (possible, of course, only in the active region, where $n'' < 0$). Figure 8 shows the amplitude of the reflection and transmission coefficients for a slab with $kd = 3$. In the part of the complex n plane of the illustration, three resonances can be observed.

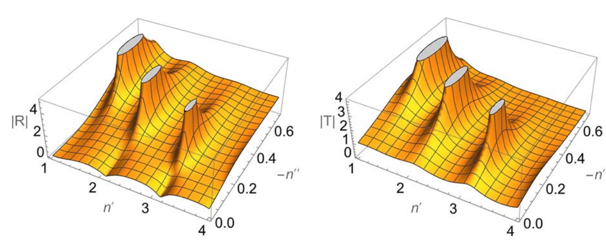

Figure 8. The absolute value of the reflection and transmission coefficients for an active slab with optical thickness $kd = 3$, as a function of the complex refractive index $n = n' - \mathrm{j}\,n''$ of the slab.

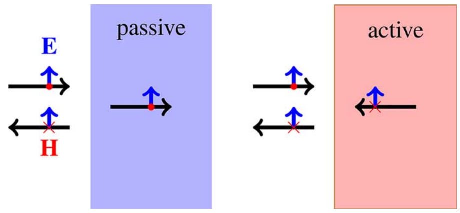

Figure 9. The propagation of reflected and transmitted plane waves in passive and active media, excited by an incident wave into the $+z$ direction. Note the direction of the magnetic field.

The resonance condition reads

$$e^{2jnkd} = \left(\frac{n-1}{n+1}\right)^2 \tag{10}$$

showing the poles of the reflection and transmission coefficients. For $kd = 3$, the three first ones are $n = 1.310 + j0.475$, $2.152 + j0.325$, and $3.158 + j0.214$. The position of the poles also has its effects in the transmission coefficient T for the lossless case ($n'' = 0$): the maximum transmission happens for integer multiples of $n'kd/\pi$, being $n' = 1.047, 2.094$ and 3.142 (close to the poles), as shown in the figure.

3.3 Reflection From Active Half-Space

A final observation from the previous analysis carries pedagogical value. What is the reflection coefficient for a normally incident plane wave that hits the boundary of an active dielectric from free space? The classical textbook solution [7] for passive (lossless and dissipative) media is, in terms of the impedances in vacuum η_0 and the half-space ($\eta = \eta_0/n$),

$$R_{\text{passive}} = \frac{\eta - \eta_0}{\eta + \eta_0} = \frac{1-n}{1+n} \tag{11}$$

This works well for reflection from passive media, which include the classes of lossless ($n'' = 0$) and dissipative ($n'' > 0$) materials. However, taking a gainy medium, for example, $n = 2 + j$, we arrive at $R_{\text{passive}} = -0.4 - j0.2$, with an absolute value less than unity. However, for an active slab with increasing thickness d, we see from (4) that the reflection magnitude is always larger than unity, as it should be for an active half-space because it creates power.

Hence, the reflection expression for the passive medium does not hold. A correct reflection coefficient comes from the observation that the *transmitted* wave is propagating into the $-z$ direction (as can be inferred, for example, from the behavior of the Poynting vector in Figure 6). The coefficient A (5) vanishes with increasing d. So, the correct view for the reflection constellation is as shown in Figure 9. Thus, in the boundary conditions, the magnetic field sign has to be carefully accounted for. The usual argumentation for passive media states the continuity of the (tangential) electric field at the boundary being $1 + R = T$ and the one for the magnetic field $1/\eta_0 - R/\eta_0 = T/\eta$, leading to (11).

However, according to Figure 9, the magnetic field continuity has to be written as $1/\eta_0 - R/\eta_0 = -T/\eta$; therefore, the reflection coefficient becomes

$$R_{\text{active}} = \frac{\eta + \eta_0}{\eta - \eta_0} = \frac{1+n}{1-n} \tag{12}$$

that agrees with $R_{\text{active}} = -2 + j$ for $n = 2 + j$.

4. Conclusion

The simple problem of an electromagnetic plane wave with normal incidence on a dielectric slab has been shown to exhibit interesting and intricate characteristics. The usual textbook analysis of passive materials has to be carefully revisited when active (gainy) materials are involved. This has been supported by numerical examples discussing the field behavior and power propagation within and outside the slab. A generalization of the analysis to single-negative materials (where the real part of the permittivity is negative) is straightforward but care to be taken so that the branch of the square root of the permittivity preserves the active character of the material. However, because the analysis was restricted to nonmagnetic materials (the relative permeability is unity), it is not applicable to double-negative media for which both the permittivity and permeability are negatively valued.

5. References

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