

Singular Behavior Analysis of Green's Functions in Uniaxial Media Using Ordinary Differential Equations

Elias Le Boudec, Nicolas Mora, Juan R. Mosig, Farhad Rachidi, Marcos Rubinstein, and Felix Vega

Abstract – We derive the electric and magnetic Green's functions in homogeneous dielectric uniaxial media by casting the problem as ordinary differential equations. These Green's functions feature an apparent singularity on the medium's distinguished axis. Using rigorous mathematical models of the fields, thanks to Schwartz distributions, we show that Green's functions are only singular at the origin. In doing so, we give a practical criterion to determine the singular behavior of derivatives of singular functions. Next, we provide the value of Green's functions along the distinguished axis. Finally, we outline possible generalizations of the approach to homogeneous biaxial media.

1. Introduction

Green's functions are central theoretical and practical tools for solving partial differential equations, particularly in electromagnetics (e.g., the method of moments [1] or the multipole expansion [2]). While these functions are well understood in isotropic and homogeneous media, issues remain with anisotropic media, where the propagation of waves depends on the direction of travel. Such media are present in nature (e.g., calcite at optical frequencies [3]) or can be synthesized by stacking electrically thin isotropic dielectric sheets [4]. Applications of artificial uniaxial media include subdiffraction imaging (i.e., super-resolution) [5].

In this paper, we focus on anisotropic media that are homogeneous, uniaxial, and dielectric, that

is, whose relative electric permittivity is given by the diagonal tensor $\bar{\bar{\epsilon}} = \epsilon_x \mathbf{e}_x \mathbf{e}_x^\top + \epsilon_y \mathbf{e}_y \mathbf{e}_y^\top + \epsilon_{zz} \mathbf{e}_z \mathbf{e}_z^\top$, and the magnetic permeability is that of vacuum. ϵ_{zz} is assumed to be a positive real number. We assume the coordinate axes have been rotated to align the z -axis with the medium's distinguished axis. Green's function for uniaxial media has been given in a coordinate-free formulation in [6] and later generalized to magnetic media in [7].

As highlighted in the latter reference, Green's functions along the distinguished axis cannot be directly evaluated due to the inclusion of a term in $1/\rho^2$ (where $\rho = \sqrt{x^2 + y^2} = 0$, $z \neq 0$ is the distance to the distinguished axis). This spurious singularity contradicts the intuition that Green's functions should only be singular in the source region. Of note, a spectral-domain transmission-line equivalent model together with Hankel transforms [8] is free from such a singularity. This issue has also been noted in [9, 10], where it was circumvented using a limiting process. However, the issue remains as this limiting process does not work in general. Indeed, take, for example, the case of Green's function for the two-dimensional Laplace equation, $(2\pi)^{-1} \log(\rho)$. Outside the origin, the Laplacian of this function is zero. In turn, its limit as $\rho \rightarrow 0$ is (trivially) also zero. However, from the definition, the Laplacian of Green's function is the Dirac δ distribution; in other words, the existence of the limit of Green's function as ρ tends to zero does not necessarily mean that Green's function is not singular.

Notably, proving that the wave operator is hypoelliptic [11] would be sufficient to show that a singularity does not exist. A differential operator is said to be hypoelliptic in an open set if every smooth source yields a smooth field. Indeed, the Dirac δ is smooth in the open set $\mathbb{R}^3 \setminus \{\mathbf{0}\}$.

In what follows, we introduce an alternative derivation of the electric and magnetic Green's functions for dielectric uniaxial media, emphasizing the distinguished axis. We show that, as expected from the physics, there is no singularity on the distinguished axis apart from the origin. Moreover, we explicitly compute Green's functions along this axis. The derivation involves the inversion of the tensor-valued wave operator, thanks to ordinary (instead of partial) differential equations. This approach might conclusively help with cases where closed forms of

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Elias Le Boudec, Juan R. Mosig, and Farhad Rachidi are with the Ecole polytechnique fédérale de Lausanne (EPFL), Station 11, 1015 Lausanne, Switzerland; e-mail: elias.leboudec@epfl.ch, juan.mosig@epfl.ch, farhad.rachidi@epfl.ch.

Nicolas Mora is with the Universidad Nacional de Colombia, Facultad de Ingeniería, Departamento de Ingeniería Eléctrica y Electrónica, Ciudad Universitaria, Edificio 411 Oficina 210, Bogotá, Colombia; e-mail: nmorap@unal.edu.co.

Marcos Rubinstein is with the University of Applied Sciences and Arts Western Switzerland, Route de Cheseaux 1, 1400 Yverdon-les-Bains, Switzerland; e-mail: marcos.rubinstein@heig-vd.ch.

Felix Vega is with the Technology Innovation Institute (TII), P.O. Box 9639, Yas Island, Abu Dhabi, United Arab Emirates; e-mail: felix.vega@tii.ae.

Green's function are unavailable, such as for biaxial media. Finally, the used approach can be applied or generalized to numerous cases where taking distributional derivatives of singular functions might introduce Dirac-like singularities.

2. Obtaining Green's Functions Through Ordinary Differential Equations

We start with Maxwell's equations in linear, homogeneous, and anisotropic media. For dielectric media, where the relative permittivity tensor is given by the tensor $\bar{\bar{\epsilon}}$ and the permeability is that of vacuum, we find that the electric and magnetic Green's functions must satisfy

$$\nabla \times [\nabla \times \mathbf{G}_e(\mathbf{r})] - k_0^2 \bar{\bar{\epsilon}} \mathbf{G}_e(\mathbf{r}) = \delta^3(\mathbf{r}) \mathbf{I} \quad (1)$$

$$\nabla \times \left\{ \bar{\bar{\epsilon}}^{-1} [\nabla \times \mathbf{G}_m(\mathbf{r})] \right\} - k_0^2 \mathbf{G}_m(\mathbf{r}) = \delta^3(\mathbf{r}) \mathbf{I} \quad (2)$$

where $\nabla^\top = [\partial/\partial x \ \partial/\partial y \ \partial/\partial z]$, $\mathbf{r}^\top = [x \ y \ z]$, k_0 is the free-space wave number, δ^3 is the three-dimensional Dirac δ distribution, and \mathbf{I} is the identity matrix. We perform all computations in the frequency domain at frequency ω and keep the frequency dependence implicit. The electromagnetic fields are obtained as

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu_0 \iiint G_e(\mathbf{r} - \mathbf{r}') \mathbf{J}(\mathbf{r}') d^3\mathbf{r} \quad (3)$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 \iiint G_m(\mathbf{r} - \mathbf{r}') \nabla \times \left[\bar{\bar{\epsilon}}^{-1} \mathbf{J}(\mathbf{r}') \right] d^3\mathbf{r} \quad (4)$$

The magnetic case (i.e., when the permeability is a second-order tensor and the permittivity is a scalar) can be obtained by duality. For the sake of brevity, we omit the case of general uniaxial media, where both the permittivity and the permeability are tensorial.

As in other derivations [6, 7], we perform a three-dimensional spatial Fourier transform with coordinates $\mathbf{k}^\top = [k_x \ k_y \ k_z]$ of (1) and (2) to find

$$\mathbf{W}_e(\mathbf{k}) \mathbf{G}_e(\mathbf{k}) := \mathbf{k} \times [\mathbf{k} \times \mathbf{G}_e(\mathbf{k})] + k_0^2 \bar{\bar{\epsilon}} \mathbf{G}_e(\mathbf{k}) = -\mathbf{I} \quad (5)$$

$$\begin{aligned} \mathbf{W}_m(\mathbf{k}) \mathbf{G}_m(\mathbf{k}) &:= \mathbf{k} \times \left\{ \bar{\bar{\epsilon}}^{-1} [\mathbf{k} \times \mathbf{G}_m(\mathbf{k})] \right\} \\ &+ k_0^2 \mathbf{G}_m(\mathbf{k}) = -\mathbf{I} \end{aligned} \quad (6)$$

In what follows, we focus on the salient parts of the derivation of $\mathbf{G}_m(\mathbf{r})$ alone, which corresponds to the method introduced in [12] for \mathbf{G}_e . Using Mathematica's "Apart" implementation of partial fraction decomposition, we find

$$\begin{aligned} \mathbf{W}_m(\mathbf{k})^{-1} &= (k_0^2 - \mathbf{k}^\top \mathbf{k}) (\mathbf{I} - \mathbf{e}_z \mathbf{e}_z^\top) h_m(\mathbf{k}) \\ &+ \left(\frac{k_z}{k_0} \right)^2 (\mathbf{k} - k_z \mathbf{e}_z) (\mathbf{k} - k_z \mathbf{e}_z)^\top h_m(\mathbf{k}) \\ &+ (\mathbf{k}_\epsilon \times \mathbf{e}_z) (\mathbf{k}_\epsilon \times \mathbf{e}_z)^\top h_m(\mathbf{k}_\epsilon) \\ &+ \frac{1}{k_0^2} \left[k_z \mathbf{k} \mathbf{e}_z^\top + k_z \mathbf{e}_z \mathbf{k}^\top - (k_0^2 + k_z^2) \mathbf{e}_z \mathbf{e}_z^\top \right] g(\mathbf{k}) \end{aligned} \quad (7)$$

where $\mathbf{k}_\epsilon = [k_x/\sqrt{\epsilon_{zz}} \ k_y/\sqrt{\epsilon_{zz}} \ k_z]$ corresponds to the elliptical part of the dispersion relation and

$$g(\mathbf{k}) = \frac{1}{|\mathbf{k}|^2 - k_0^2} \quad (8)$$

$$h_m(\mathbf{k}) = \frac{1}{k_z^2 - k_0^2} g(\mathbf{k}) \quad (9)$$

By using the scaling property of the Fourier transform, (7) can be transformed into space-domain expressions involving $g(\mathbf{r})$ and $h_m(\mathbf{r})$:

$$\begin{aligned} \mathbf{G}_m(\mathbf{r}) &= -(\nabla^\top \nabla + k_0^2) (\mathbf{I} - \mathbf{e}_z \mathbf{e}_z^\top) h_m(\mathbf{r}) \\ &+ \left(\frac{1}{k_0} \frac{\partial}{\partial z} \right)^2 \left(\nabla - \frac{\partial}{\partial z} \mathbf{e}_z \right) \left(\nabla - \frac{\partial}{\partial z} \mathbf{e}_z \right)^\top h_m(\mathbf{r}) \\ &+ (\nabla \times \mathbf{e}_z) (\nabla \times \mathbf{e}_z)^\top h_m(\mathbf{r}_\epsilon) \\ &+ \frac{1}{k_0^2} \left[\nabla \mathbf{e}_z^\top \frac{\partial}{\partial z} + \mathbf{e}_z^\top \nabla \frac{\partial}{\partial z} - \left(\frac{\partial^2}{\partial z^2} - k_0^2 \right) \mathbf{e}_z \mathbf{e}_z^\top \right] g(\mathbf{r}) \end{aligned} \quad (10)$$

where $\mathbf{r}_\epsilon^\top = [\sqrt{\epsilon_{zz}}x \ \sqrt{\epsilon_{zz}}y \ z]$. The distribution g is known to be Green's function for the scalar wave equation (e.g., [7, eq. (44)] with the opposite Fourier transform convention), while h_m satisfies

$$\left(\frac{\partial^2}{\partial z^2} + k_0^2 \right) h_m(\mathbf{r}) = -g(\mathbf{r}) \quad (11)$$

To ensure that (2) is satisfied, the distribution h_m must also satisfy a homogeneity constraint

$$(\nabla^\top \nabla + k_0^2) h_m(\mathbf{r}) = H(\mathbf{r}) \quad (12)$$

where H is a distribution such that $H(\sqrt{\epsilon_{zz}}x, \sqrt{\epsilon_{zz}}y, z) = \frac{1}{\epsilon_{zz}} H(x, y, z)$. In other words, H is homogeneous of degree -2 in the first two variables. Combining (11) and (12), an immediate consequence is that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) h_m(\mathbf{r}) = H(\mathbf{r}) + g(\mathbf{r}) \quad (13)$$

Choosing $H(\mathbf{r})=0$, it was shown in [12] that $h_m = -h$, where

$$h(\mathbf{r}) = \frac{e^{jk_0z} \text{Ei}[-jk_0(r+z)] + e^{-jk_0z} \text{Ei}[-jk_0(r-z)] - e^{-jk_0|z|} \log(x^2 + y^2)}{8\pi j k_0} \quad (14)$$

Ei is the exponential-integral function, defined in [13] as $\text{Ei}(z) = -\text{PV} \int_{-z}^{\infty} e^{-t}/t dt$ where PV denotes the Cauchy principal value.

To obtain the result above, it is necessary to perform distributional derivatives. Ordinary derivatives do not allow to describe the behavior of h or h_m on the distinguished axis, where $\rho = 0$. As a result, the term in $\log(x^2 + y^2)$ would be missing, and the resulting distribution would be singular on the distinguished axis.

To help compute these distributional derivatives, we introduced the following result in [12]. Let us assume that T_f is a two-dimensional distribution that is regular but has an integrable singularity on the $\rho = 0$ axis, that is, for any test function $\psi \in \mathcal{D}(\mathbb{R}^2)$ (where $\mathcal{D}(\mathbb{R}^2)$ is the set of two-dimensional smooth functions with compact support; for Fourier analysis—e.g., tempered distributions—this set might be replaced by the Schwartz space of “rapidly decreasing” smooth functions [14]).

$$\langle T_f, \psi \rangle = \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \int_{\varepsilon}^{\infty} f(\rho, \phi) \psi(\rho, \phi) \rho d\rho d\phi \quad (15)$$

where f is an integrable function differentiable outside the distinguished axis. In Appendix A, we show that

$$\left\langle \frac{\partial T_f}{\partial x}, \psi \right\rangle = \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \{ \rho f(\rho, \phi) \psi(\rho, \phi) \cos(\phi) |_{\rho=\varepsilon} + \int_{\varepsilon}^{\infty} \frac{\partial f(\rho, \phi)}{\partial x} \psi(\rho, \phi) \rho d\rho \} d\phi \quad (16)$$

If the limits of the two terms exist independently, we obtain

$$\frac{\partial T_f}{\partial x} = \delta^2 \lim_{\rho \rightarrow 0} \underbrace{\int_0^{2\pi} \rho f(\rho, \phi) \cos(\phi) d\phi}_{\text{constant}} + T_{\partial f/\partial x} \quad (17)$$

where $T_{\partial f/\partial x}$ is the distribution associated with the classical derivative of the function f outside the singularity $\rho = 0$, and δ^2 is the two-dimensional xy -dependent Dirac δ distribution.

To illustrate, we will show that $f(\rho, \phi) = \log(\rho)$ is Green’s function for the two-dimensional Laplace equation. Indeed, the singular part of $\partial T_f/\partial x$ is

proportional to $\int_0^{2\pi} \log(\rho) \cos(\phi) d\phi$, which is zero. The same holds for the y derivative. In turn, the first-order distributional derivatives of f correspond to the classical ones. However, the singular part of the second-order derivative $\partial^2/\partial x^2$ is

$$\lim_{\rho \rightarrow 0} \int_0^{2\pi} \rho (\rho \cos(\phi)/\rho^2) \cos(\phi) d\phi = \pi \quad (18)$$

Repeating the same computations for y , we get $(\partial^2/\partial x^2 + \partial^2/\partial y^2) \log(\rho) = \delta^2(x, y) 2\pi$, as expected.

A careful evaluation of the singular terms in all derivatives of (10) shows that Green’s function \mathbf{G}_m features no singularity on the distinguished axis as for its electric counterpart. Moreover, the distribution \mathbf{G}_m is represented by a smooth function in all space but the distinguished axis. Because the Lebesgue measure of this axis embedded in three-dimensional space is zero, the distribution \mathbf{G}_m does not depend on its value along the distinguished axis (indeed, this value has zero contribution “under the integral sign”). Thus, we can arbitrarily fix a meaningful physical value by using the limit as $\rho \rightarrow 0$:

$$\lim_{(x,y) \rightarrow (0,0)} \mathbf{G}_{m,e}(x, y, z \neq 0) =: \mathbf{G}_{m,e}(0, 0, z) = \begin{bmatrix} f_{xy}^{m,e}(z) & 0 & 0 \\ 0 & f_{xy}^{m,e}(z) & 0 \\ 0 & 0 & f_z(z) \end{bmatrix} \quad (19)$$

with

$$f_{xy}^m(z) = -e^{-jk_0|z|} \frac{1 + jk_0|z| - k_0^2 z^2 \frac{\varepsilon_z + 1}{2}}{4k_0^2 \pi |z|^3} \quad (20)$$

$$f_{xy}^e(z) = -e^{-jk_0|z|} \frac{\varepsilon_{zz} + j\varepsilon_{zz} k_0|z| - k_0^2 z^2 \frac{\varepsilon_z + 1}{2}}{4k_0^2 \pi |z|^3} \quad (21)$$

$$f_z(z) = e^{-jk_0|z|} \frac{1 + jk_0|z|}{2k_0^2 \pi |z|^3} \quad (22)$$

The expressions above yield the fields radiated by dipoles in all polarizations along the distinguished axis, solving the issue raised in [7].

3. Conclusion

We present an alternative derivation of Green’s electric and magnetic functions in dielectric uniaxial media. The inversion of the differential operator is based on recasting the partial differential equation into ordinary differential equations. A careful evaluation of the derivatives of singular distributions shows that Green’s functions are not singular along the medium’s

distinguished axis and we give a meaningful value to the functions along this axis. We also provide a practical formula to determine the Dirac-like singularity of derivatives of singular distributions and illustrate it on the Laplace equation.

A natural extension of the presented work is the case of biaxial anisotropic media, where the permittivity or permeability tensors are allowed to possess distinct eigenvalues. Examples of such media include metamaterials (e.g., [15]). Closed-form expressions for Green's functions in biaxial media are yet to be found. The main difficulty resides in the form the dispersion relation takes (see, e.g., [16]). Indeed, while the dispersion relation of uniaxial media consists of a sphere inscribed in an ellipsoid, biaxial media must be described by the Fresnel wave surface, a particular instance of Kummer's quartic surfaces [17]. While an ellipsoid can easily undergo Fourier transforms as it results from the scaling of a sphere, such an approach is not directly possible for biaxial media.

However, decomposing the dispersion relation in two distinct surfaces is still possible. Moreover, closed-form two-dimensional parametrizations of the surfaces can be expressed thanks to Jacobi elliptic functions [17, p. 105]. Green's functions could then be obtained by analyzing the integral of plane waves following directions defined by the dispersion relation.

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Appendix A. Derivation

We look for the distributional derivatives of T_f , defined in Equation (15). By definition,

$$\left\langle \frac{\partial T_f}{\partial x}, \psi \right\rangle = - \left\langle T_f, \frac{\partial \psi}{\partial x} \right\rangle \quad (23)$$

$$= - \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \int_{\varepsilon}^{\infty} f(\rho, \phi) \frac{\partial \psi(\rho, \phi)}{\partial x} \rho d\rho d\phi \quad (24)$$

We express the x derivative in cylindrical coordinates and use integration by parts with respect to ρ to get

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \left\{ - \rho f(\rho, \phi) \psi(\rho, \phi) \cos(\phi) \Big|_{\rho=\varepsilon}^{\infty} \right. \\ \left. + \int_{\varepsilon}^{\infty} \left[\frac{\partial}{\partial \rho} \left\{ \rho f(\rho, \phi) \right\} \cos(\phi) \psi(\rho, \phi) \right. \right. \\ \left. \left. + f(\rho, \phi) \sin(\phi) \frac{\partial \psi(\rho, \phi)}{\partial \phi} \right] d\rho \right\} d\phi \end{aligned} \quad (25)$$

Using the compact support of ψ ,

$$\begin{aligned} = \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \left\{ \rho f(\rho, \phi) \psi(\rho, \phi) \cos(\phi) \Big|_{\rho=\varepsilon} \right. \\ \left. + \int_{\varepsilon}^{\infty} \left[\left\{ f(\rho, \phi) + \rho \frac{\partial f(\rho, \phi)}{\partial \rho} \right\} \cos(\phi) \psi(\rho, \phi) \right. \right. \\ \left. \left. + f(\rho, \phi) \sin(\phi) \frac{\partial \psi(\rho, \phi)}{\partial \phi} \right] d\rho \right\} d\phi \end{aligned} \quad (26)$$

Integrating by parts with respect to ϕ ,

$$\begin{aligned} = \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} \left[\rho f(\rho, \phi) \psi(\rho, \phi) \cos(\phi) \Big|_{\rho=\varepsilon} \right. \\ \left. + \int_{\varepsilon}^{\infty} \left[\left\{ f(\rho, \phi) + \rho \frac{\partial f(\rho, \phi)}{\partial \rho} \right\} \cos(\phi) \psi(\rho, \phi) \right] d\rho \right] d\phi \\ + \int_{\varepsilon}^{\infty} \left\{ \underbrace{f(\rho, \phi) \sin(\phi) \psi(\rho, \phi) \Big|_{\phi=0}^{2\pi}}_0 - \int_0^{2\pi} \left[\sin(\phi) \frac{\partial f(\rho, \phi)}{\partial \phi} \right. \right. \\ \left. \left. + \cos(\phi) f(\rho, \phi) \right] \psi(\rho, \phi) d\phi \right\} d\rho \end{aligned} \quad (27)$$

We obtain Equation (16) by canceling the second and last terms and recognizing the derivative with respect to x .