From the Ray Approximation to the Exact Method of Random Rays for Wave Scattering Problems

Bair V. Budaev and David B. Bogy

Abstract – An exact solution is presented for problems of diffraction of electromagnetic waves in media with nonconstant wave numbers. The well-known ray approximation is extended to closed form representations of wave fields regarding Feynman–Kac formulas. This approach starts as the ray method, but instead of studying the transport equation in a physical space, the problem is reduced to a transport equation in a two-sheet Riemannian manifold, which admits the exact solution by the well-known Feynman–Kac formula. The method is illustrated in a one-dimensional example with a representative wave number. Remarkably, the exact solutions proposed here appear as straightforward generalizations of the asymptotic approximations delivered by the ray method.

1. Introduction

The theory of wave propagation covers many areas, each with its own problems and methods. Thus, the theory of diffraction deals with phenomena in which electromagnetic or other waves propagate in a homogeneous medium with inclusions, such as screens, cones, and balls, which model structures, including mountains, aircraft, submarines, teeth, and bones. Another kind of problem deals with waves in inhomogeneous media, such as studies in ocean acoustics, radiophysics, atmospheric sciences, and ultrasound and X-ray diagnostics. Problems of this type may include equations with smoothly varying parameters, which often admit asymptotic solutions based on the *ray approximation* [1–4].

We present an extension of the ray approximation to a method providing closed form exact solutions of the Helmholtz equation with a nonconstant wave number. This method uses probabilistic Feynman–Kac formulas that deliver exact solutions of a broad class of partial differential equations. It is shown that the obtained exact solutions appear as straightforward generalizations of the ray method approximation.

2. Preliminaries

Our approach to wave propagation in inhomogeneous media starts from seeking the solution of the Helmholtz equation

$$\nabla^2\phi(x)+\Omega^2 L(x)\phi(x)=0, \qquad \Omega=\text{const} \tag{1}$$

in the form

$$\phi(x)=u_+(x)\mathrm{e}^{\mathrm{i}\Omega S(x)}+u_-(x)\mathrm{e}^{-\mathrm{i}\Omega S(x)} \tag{2}$$

where $S(x)$ obeys the eikonal

$$|\,\boldsymbol{\nabla} S(x)\,|^2=L(x) \tag{3}$$

and $u_\pm(x)$ obey the transport equations

$$\begin{aligned}&\frac{\mathrm{i}}{2\Omega}\nabla^2u_+(x)-g(x)\boldsymbol{e}(x)\boldsymbol{\nabla}u_+(x)-b(x)u_+(x)=0\\&\frac{\mathrm{i}}{2\Omega}\nabla^2u_-(x)+g(x)\boldsymbol{e}(x)\boldsymbol{\nabla}u_-(x)+b(x)u_-(x)=0\end{aligned} \tag{4}$$

with the coefficients

$$\begin{aligned}&g(x)=\pm\sqrt{L(x)},\\&\boldsymbol{e}(x)=\frac{\boldsymbol{\nabla}S(x)}{g(x)},\quad b(x)=\frac{1}{2}\operatorname{div}[g(x)\boldsymbol{e}(x)]\end{aligned} \tag{5}$$

The eikonal (3) can be handled by a Hamiltonian method, whose foundations and fast implementations are discussed in [5, 6], respectively. For a given point x_0, the eikonal $S(x;x_0)$ represents the shortest time needed for the wave to travel from x_0 to x. Correspondingly, $S(x;x_0)=T$ describes the location at time T of the wave front of radiation launched from point x_0 at time $t=0$.

Huygens' principle [5] implies that wave propagation can be described both for wave fronts and for rays, which are the fastest trajectories along which waves propagate. In isotropic cases, as described by (1), rays coincide with integral lines of the vector field $\boldsymbol{\nabla}_x S(x;x_0)$ characterized by the initial position x_0 and the initial direction along a unit vector $\boldsymbol{e}$. Obviously, the rays corresponding to the initial directions $\boldsymbol{e}$ and $-\boldsymbol{e}$ form a line which is infinite in both directions as a one-dimensional waveguide.

For (4), the definition (5) of $\pm g(x)$ suggests that these functions may be treated as two branches of the square root of $L(x)$. If $L(x)$ is analytic, then $g(\chi)=\sqrt{L(\chi)}$ may be treated as a function of a point χ of a Riemannian manifold $\mathfrak{G}=\mathfrak{G}_+\cup\mathfrak{G}_-$, built from two complex planes $\mathfrak{G}_\pm$ in such a way that $g(\chi_-)=-g(\chi_+)$

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Bair V. Budaev and David B. Bogy are with the University of California, Department of Mechanical Engineering, Berkeley, California, 94720-1740, USA; e-mail: budaev@berkeley.edu, dbogy@berkeley.edu.

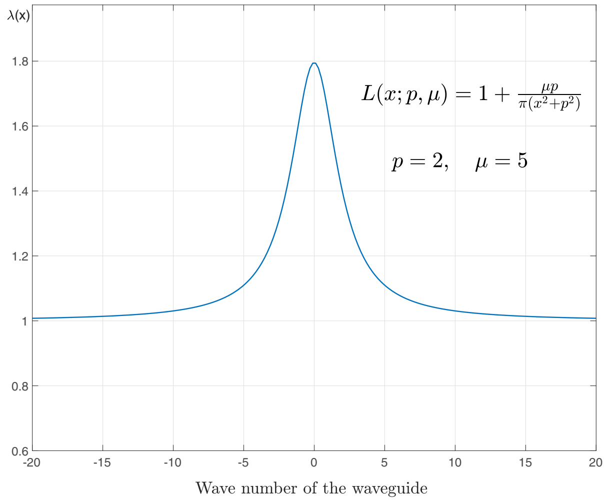

Figure 1. Profile of the wave number $L(x)$.

for $\chi_\pm \in \mathfrak{G}_\pm$. Because every $\chi \in \mathfrak{G}$ belongs to one of the sheets $\mathfrak{G}_\pm$, we agree to χ by the pairs (x,ν), where x is a point corresponding to χ in the complex space and $\nu='\pm'$ is the index of the sheet. Thus, (4) can be combined into a single equation

$$\frac{\mathrm{i}}{2\Omega}\nabla^2 U(\chi)-g(\chi)\boldsymbol{e}(\chi)\boldsymbol{\nabla}U(\chi)-b(x)U(\chi)=0 \qquad (6)$$

where $U(\chi)$ is a function of $\chi \in \mathfrak{G}$ related to $u_\pm(\chi)$ by

$$U(\chi)=u_\pm(\chi_\pm), \qquad \text{if} \quad \chi \in \mathfrak{G}_\pm \qquad (7)$$

If $U(\chi)$ is known, then a solution of (1) on $\mathfrak{G}$ is defined by

$$\phi(\chi)=U(\chi)\mathrm{e}^{\mathrm{i}\Omega S(\chi)}, \qquad \chi \in \mathfrak{G}_+ \cup \mathfrak{G}_- \qquad (8)$$

3. The Exact Solution of the Helmholtz Equation in a Nonhomogeneous Line

Let waves in an inhomogeneous waveguide be described by the Helmholtz equation

$$\phi''(x)+\Omega^2 L(x)\phi(x)=0, \qquad -\infty < x < \infty \qquad (9)$$

We assume that $L(x)$ illustrated in Figure 1, has a structure

$$L(x;p,\mu)=1+\frac{\mu p}{\pi(x^2+p^2)}, \qquad \mu>0, \quad p>0 \qquad (10)$$

whose property $\lim\limits_{p\to 0}[L(x;p,\mu)-1]=\mu\delta(x)$ makes it possible to represent any continuous $L_*(x)$ by

$$L_*(x)=\int L_*(\xi)[L(x-\xi;p,1)-1]\mathrm{d}\xi \qquad (11)$$

The coefficient $\Omega^2 L(x)$ in the (9) approaches Ω^2 as $x\to\pm\infty$. This guarantees that in the areas $x\to\pm\infty$, the field $\phi(x)$ appears as a superposition

$$\phi(x)=A\mathrm{e}^{\mathrm{i}\Omega x}+B\mathrm{e}^{-\mathrm{i}\Omega x} \qquad (12)$$

of waves propagating in the directions $x\to\pm\infty$, respectively. Therefore, the field excited by the wave $\mathrm{e}^{\mathrm{i}\Omega x}$ arriving from $x=-\infty$ must have the asymptotes

$$\phi(x)=\begin{cases}\mathrm{e}^{\mathrm{i}\Omega x}+R\mathrm{e}^{-\mathrm{i}\Omega x}, & x\to-\infty\\ T\mathrm{e}^{\mathrm{i}\Omega x}+0\mathrm{e}^{-\mathrm{i}\Omega x}, & x\to+\infty\end{cases} \qquad (13)$$

where the term $0\mathrm{e}^{-\mathrm{i}\Omega x}$ means that there are no waves arriving from $x=+\infty$, the term $\mathrm{e}^{\mathrm{i}\Omega x}$ describes the incident wave, and the terms $R\mathrm{e}^{-\mathrm{i}\Omega x}$ and $T\mathrm{e}^{\mathrm{i}\Omega x}$ describe reflected and transmitted waves, characterized by indefinite reflection and transmission coefficients R and T, whose computations make an important part of the problem.

As suggested in Section 2, we introduce a two-sheet Riemannian manifold $\mathfrak{G}=\mathfrak{G}_+\cup\mathfrak{G}_-$ of the wave number (10) and agree that χ is a point on $\mathfrak{G}$, and symbols $\chi_\pm$ denote points on sheets $\mathfrak{G}_\pm$, as shown in Figure 2. Thus, (10) can be transformed to the form

$$g(\chi)\equiv\sqrt{L(\chi)}=\sqrt{\frac{\chi^2+a^2}{\chi^2+p^2}}, \qquad a^2=p^2+\frac{\mu p}{\pi} \qquad (14)$$

Let sheets $\mathfrak{G}_+$ and $\mathfrak{G}_-$ be the complex planes cut along the intervals $(\mathrm{i}p,\mathrm{i}a)$ and $(-\mathrm{i}p,-\mathrm{i}a)$ and let the indices of the sheets be assigned by the rule

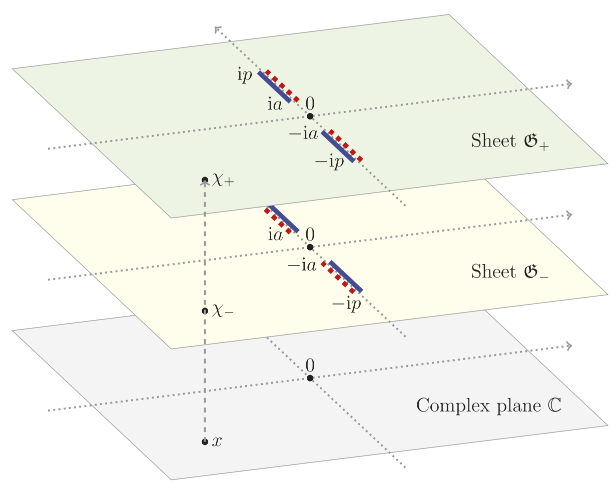

Figure 2. Two-sheet Riemannian manifold $\mathfrak{G}=\mathfrak{G}_+\cup\mathfrak{G}_-$. The manifold $\mathfrak{G}$ consists of two sheets $\mathfrak{G}_\pm$ stacked above the complex plane $\mathbb{C}$. Points on $\mathfrak{G}_\pm$ located above $x\in\mathbb{C}$ are denoted by $\chi_\pm$, respectively. Each sheet has cuts $(\mathrm{i}a,\mathrm{i}p)$ and $(-\mathrm{i}a,-\mathrm{i}p)$. The dotted (solid) sides of the cuts located above each other are connected, which make possible continuous paths between different sheets.

$$\mathfrak{G}_\pm: \qquad g(\chi) \to \pm 1, \qquad \text{as} \quad |\chi| \to \infty \tag{15}$$

Symbols $\chi = \infty_\pm$ denote infinite points on $\mathfrak{G}_\pm$ so that $\chi = \infty_\pm$ means that $\chi \in \mathfrak{G}_\pm$ and $|\chi| = \infty$. Symbols $\pm\infty_+$ and $\pm\infty_-$ denote *lines* $\mathrm{Re}(\chi) = \pm\infty$ on sheets $\mathfrak{G}_\pm$.

According to (6)–(8), the solution of the problem (9)–(13) considered on $\mathfrak{G}$ is represented by

$$\phi(\chi) = U(\chi)\mathrm{e}^{\mathrm{i}\Omega S(\chi)}, \quad S(\chi) = \int_0^\chi g(\xi)\,\mathrm{d}\xi \tag{16}$$

and $U(\chi)$ is the solution of the transport equation

$$\frac{\mathrm{i}}{2\Omega} U''(\chi) - g(\chi)U'(\chi) - \frac{1}{2} g'(\chi)U(\chi) = 0 \tag{17}$$

considered in the manifold $\mathfrak{G}$ with conditions at infinity

$$U(\chi) = \begin{cases} 1, & \chi = -\infty_+ \subset \mathfrak{G}_+ \\ 0, & \chi = +\infty_- \subset \mathfrak{G}_- \end{cases} \tag{18}$$

The solution of (17) with conditions at infinity (18) is delivered by the Feynman–Kac formula [7]

$$U(\chi) = \left\langle U(\xi_\tau)\exp\left(-\frac{1}{2}\int_0^\tau g'(\xi_t)\mathrm{d}t\right)\right\rangle \tag{19}$$

where the brackets $\langle\cdot\rangle$ denote the averaging over all trajectories of the random motion ξ_t that runs on $\mathfrak{G}$ under the control of Ito's stochastic differential equation [7]

$$\mathrm{d}\xi_t = \sqrt{\frac{\mathrm{i}}{\Omega}}\mathrm{d}w_t - g(\xi_t)\mathrm{d}t, \qquad \xi_0 = \chi \tag{20}$$

where w_t is the Brownian motion stopped at the time τ, when ξ_t reaches one of the *exit* lines $\xi_\tau = -\infty_+$ or $\xi_\tau = +\infty_-$, where the *boundary* conditions (18) are specified.

To verify conditions (18), we recall that χ is the observation point and ξ_t is the trajectory of the random motion launched from $\xi_0 = \chi$ and stopped at ξ_τ on the boundary, where $U(\xi_\tau)$ is set by the boundary conditions (18). These definitions guarantee that in a bounded domain formula, (19) satisfies boundary conditions (18), because if χ is located on the boundary, then $\xi_\tau = \chi$, and (19) reduces to the identity $U(\chi) = \langle U(\chi)\rangle$. The problem of our interest is formulated on an unbounded manifold with boundary conditions at infinities $-\infty_+$ and $+\infty_-$, which can be approximated by straight lines $-I_+$ and $+I_-$, defined by

$$\begin{aligned} \pm I_+ &= \{\chi \in \mathfrak{G}_+ : \quad \mathrm{Re}(\chi) = \pm M\} \\ \pm I_- &= \{\chi \in \mathfrak{G}_- : \quad \mathrm{Re}(\chi) = \pm M\} \end{aligned} \tag{21}$$

where $M \gg 1$ is selected to satisfy conditions: $|g'(\chi)|$ is negligible in the domain $|\mathrm{Re}(\chi)| > M/2$; and any path ξ_t started from area $|\mathrm{Re}(\chi)| > L/2$ on either sheet has negligible chances to change the sheet.

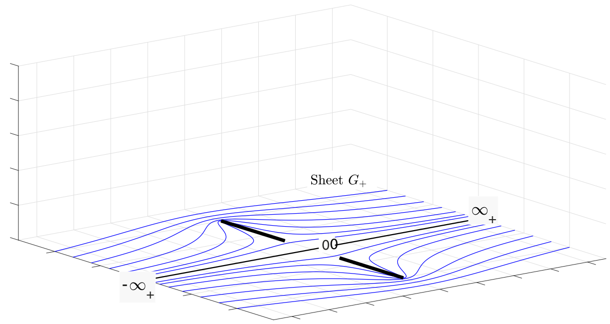

Figure 3. Deterministic rays in the ray method. This diagram shows trajectories defined by (20) on the upper sheet $\mathfrak{G}_+$ for infinite frequency Ω. Trajectories start from equally spaced points on the line $\mathrm{Re}(\chi) = M \gg 1$ on sheet $\mathfrak{G}_+$. Because $\Omega = \infty$, all trajectories are deterministic and stay within $\mathfrak{G}_+$.

4. Comments to the Feynman–Kac Solution

First, we verify that for $\Omega = \infty$, the solution (19) coincides with the standard ray method asymptote. In this case, the stochastic (20) reduces to a deterministic equation $\mathrm{d}\xi_t = -g(\xi_t)\mathrm{d}t$, where $g(\chi)$ has the properties

$$\mathrm{Re}\ g(\chi) \geq 0, \quad \text{if} \quad \chi \in \mathfrak{G}_+ \tag{22}$$

$$\mathrm{Re}\ g(\chi) \leq 0, \quad \text{if} \quad \chi \in \mathfrak{G}_- \tag{23}$$

$$\mathrm{Re}\ g(\chi) = 0, \quad \text{if} \quad \chi \in (\mathrm{i}p, \mathrm{i}a) \cup (-\mathrm{i}a, -\mathrm{i}p) \tag{24}$$

which secure the following features of the paths ξ_t: property (22) guarantees that on $\mathfrak{G}_+$ points ξ_t drift toward $-\infty$; similarly, (23) guarantees that on $\mathfrak{G}_-$, these points drift toward $-\infty$; and (24) guarantees that trajectories of ξ_t never intersect the cuts $\pm(\mathrm{i}a, \mathrm{i}p)$ and never move between sheets $\mathfrak{G}_+$ and $\mathfrak{G}_-$, as seen in Figure 3. This figure shows how deterministic trajectories launched from the remote points $x = M$, $M \pm \mathrm{i}, M \pm 2\mathrm{i} \ldots$, on the right-hand side of sheet $\mathfrak{G}_+$, proceed. The path launched from the real axis remains on this axis; all other paths go either between the cuts $\pm(\mathrm{i}a, \mathrm{i}p)$ or go around these cuts, and no one trajectory penetrates to sheet $\mathfrak{G}_-$.

Trajectories of ξ_t radically change when Ω is finite. In this case, each step of the motion ξ_t controlled by (20) includes a random shift $\sqrt{\frac{\mathrm{i}}{\Omega}}\mathrm{d}w_t$, which makes it possible for ξ_t to cross one of the cuts $\pm(\mathrm{i}a, \mathrm{i}p)$ and move to another sheet. This behavior of trajectories is illustrated in Figure 4. All trajectories start from the same positions on $\mathfrak{G}_+$, as illustrated in Figure 3. In this case, no one path follows along the real axis, some paths squeeze between the cuts, and other paths go around the cuts and remain on $\mathfrak{G}_+$. However, some paths, as shown by the red lines, cross the cuts and move to sheet $\mathfrak{G}_-$, as illustrated by dashed lines. This transition of random rays to another sheet corresponds to the formation of scattered waves.

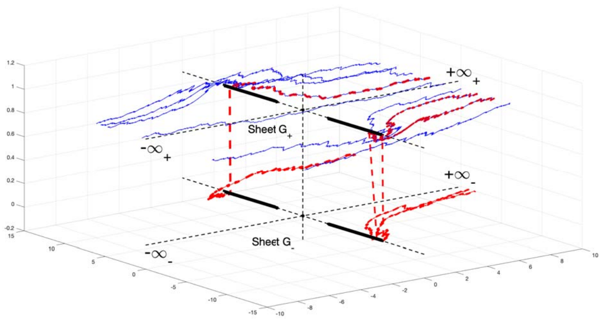

Figure 4. Random rays in a two-sheet surface: sheet $\mathfrak{G}_+$ and sheet $\mathfrak{G}_-$. This diagram shows trajectories defined by (20) in the two-sheet manifold $\mathfrak{G}$ for finite frequencies Ω. Trajectories start from equally spaced points on the line $\mathrm{Re}(\chi) = M \gg 1$ on sheet $\mathfrak{G}_+$. Because Ω is finite, paths have random steps and may penetrate to sheet $\mathfrak{G}_-$ and change the direction of motion.

It is easy to see that the procedure described in conjunction with (21) secures conditions at infinity (18). As discussed at the end of Section 3, for the satisfaction of these conditions, it is necessary that the random paths (20) launched from point $\chi \ll -1$ on sheet $\mathfrak{G}_+$ move toward $-\infty$, while paths launched from $\chi \gg 1$ on sheet $\mathfrak{G}_-$ move toward ∞, as shown in Figure 5.

5. The Ray Approximation Is a Special Case of the Exact Random Ray Method

As previously mentioned, the exact solution of the equation

$$\phi''(x) + \Omega^2 L(x)\phi(x) = 0, \qquad -\infty < x < \infty \tag{25}$$

with conditions at infinity

$$\phi(x) = \begin{cases} \mathrm{e}^{\mathrm{i}\Omega x} + R\mathrm{e}^{-\mathrm{i}\Omega x}, & x \to -\infty \\ T\mathrm{e}^{\mathrm{i}\Omega x}, & x \to +\infty \end{cases} \tag{26}$$

was represented by

$$\phi(x) = U(\chi_+)\mathrm{e}^{\mathrm{i}\Omega S(\chi_+)} + U(\chi_-)\mathrm{e}^{\mathrm{i}\Omega S(\chi_-)} \tag{27}$$

where $S(\chi)$ is the eikonal, defined in the two-sheet Riemannian manifold $\mathfrak{G}$ of $\sqrt{L(\chi)}$, and $\chi_\pm$ are points on sheets $\mathfrak{G}_\pm$, associated with the physical real-valued coordinate x. As for the amplitudes $U(\chi_\pm)$, they are defined as the averages

$$U(\chi_\pm) = \left\langle U(\xi_\tau)\exp\left(-\frac{1}{2}\int_0^\tau g'(\xi_s)\mathrm{d}s\right)\right\rangle \tag{28}$$

over all trajectories of a random motion ξ_t controlled by a stochastic differential equation

$$\mathrm{d}\xi_t = -g(\xi_t)\mathrm{d}t + \sqrt{\frac{\mathrm{i}}{\Omega}}\mathrm{d}w_t, \qquad \xi_0 = \chi_\pm \tag{29}$$

that are stopped at the exit time τ, defined as the first time when ξ_t hits the boundary of a predefined strip $|\mathrm{Re}(\xi)| = M \gg 1$, where $g(\xi) = 1 + o(1)$.

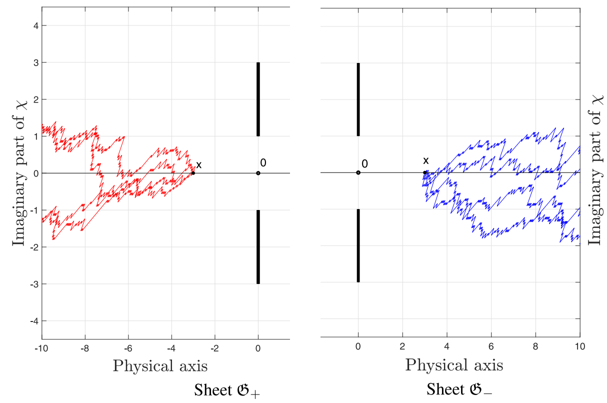

Figure 5. Satisfaction of the conditions at infinity (18) in sheets $\mathfrak{G}_+$ and $\mathfrak{G}_-$.

As $\Omega \to \infty$, (29) reduces to $\mathrm{d}\xi_t = -g(\xi_t)\mathrm{d}t$, implying that the exact solution of (27) and (28) of the scattering problem (25) and (26) reduces to the ray approximation. This justifies the claim that the method of random rays presented here extends the ray approximation to the exact solution of the wave scattering problem.

The method of random rays providing closed form expressions for wave fields in inhomogeneous media was illustrated previously in a one-dimensional example with a variable wave number described by an elementary analytic function (10). The choice of the one-dimensional model made possible visual illustrations of rays in a complex plane, which are essential for the proposed method. However, the core of the method described by (6)–(8) remains valid for spaces of any dimension, opening a broad range of applications to a wide range of problems of wave propagation and quantum mechanics, related to the Helmholtz equations.

Because the proposed method uses stochastic processes, it seems natural to extend it to problems of wave propagation in random media [8]. Also, the close connection between the Helmholtz and Schrödinger equations suggest that the proposed method may find applications in quantum mechanics.

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7. References

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