Plane-Wave Diffraction by a Slit Formed by Two Semi-Infinite Parallel-Plate Waveguides—Part II: The Case of $E$ Polarization

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Abstract — The diffraction by a slit formed by two semi-infinite parallel-plate waveguides is rigorously analyzed using the Wiener–Hopf technique for the $E$-polarized plane-wave incidence. The method similar to that developed in Part I of this two-part article, exact and high-frequency asymptotic solutions are both obtained. Numerical examples of the far-field intensity are presented, and scattering characteristics of the waveguide slit are discussed. Some comparisons with our results for $H$ polarization are also given.

1. Introduction

The analysis of electromagnetic scattering by slits formed by parallel-plate waveguides (i.e., tandem slits) is important in electromagnetic theory, as well as microwave and optical instrumentation. There are several important articles analyzing diffraction problems related to tandem slits [1–5]. Among these, [3] is most relevant to this two-part article and analyzed the plane-wave diffraction by a perfectly conducting tandem slit, with the aid of the Wiener–Hopf technique, leading to a high-frequency asymptotic solution.

In Part I of this two-part article [6], we considered the same waveguide slit geometry as in [3] and rigorously analyzed the $H$-polarized plane wave diffraction using the Wiener–Hopf technique. As a result, we successfully obtained a novel high-frequency solution to the waveguide slit problem in the form of a complete asymptotic series by using rigorous asymptotics [7, 8] developed by the authors for the analysis of modified Wiener–Hopf geometries. Note that our asymptotic solution incorporates all the higher order diffraction effects; hence, it is valid, even for slit width comparable to the incident wavelength.

This article serves as the second part of this two-part article, and we analyze the $E$-polarized plane wave diffraction by the same, waveguide slit geometry as considered in Part I. The method of solution is based on the Wiener–Hopf technique, and we apply a method similar to that developed for the analysis of $H$ polarization. The Wiener–Hopf analysis leads first to an exact (formal) solution and then to a high-frequency asymptotic solution. As has been previously emphasized, the latter solution contains all the higher order diffraction terms and, hence, serves as a strong solution. The scattered field is subsequently evaluated, resulting in transverse electric modes inside the waveguides and a far-field expression outside the waveguides. Numerical examples are presented for various physical parameters, and the scattering characteristics of the waveguide slit are discussed. Some comparisons with the results for $H$ polarization [6] are given. The time factor is assumed to be $\exp(-i\omega t)$ and suppressed throughout the article.

2. Formulation of the Problem

The geometry of the waveguide slit is shown in Figure 1, where the plates of the waveguides are infinitely thin and uniform in the $y$ direction, and $\phi^i$ denotes the $E$-polarized plane wave. Let the total electric field $\phi(x, z) = E_i(x, z)$ be defined by

$$\phi(x, z) = \phi^i(x, z) + \phi(x, z) + R\phi^i(x, z), \quad x > b$$

where $\phi^i(x, z)$ is the incident field defined by

$$\phi^i(x, z) = e^{-ik(x \sin \theta_0 + z \cos \theta_0)}, \quad 0 < \theta_0 < \pi/2$$

with $k = \omega(\varepsilon_0 \mu_0)^{1/2}$ being the free-space wavenumber. In (1), $R\phi^i(x, z)$ denotes the reflected field, where

$$\phi^r(x, z) = e^{ik(x \sin \theta_0 - z \cos \theta_0)}$$

and $R = -e^{-2ik \sin \theta_0}$ is the reflection coefficient. In the following analysis, we assume that the medium is slightly lossy as in $k = k_1 + ik_2, 0 < k_1 \ll k_2$.

We define the Fourier transform of $\phi(x, z)$ with respect to $z$ as

$$\Phi(x, \zeta) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \phi(x, z)e^{i\zeta z}dz, \quad \zeta = \sigma + i\tau$$

We also define the Fourier integrals as

$$\Phi_{\pm}(x, \zeta) = (2\pi)^{-1/2} \int_{\pm\infty}^{\pm\infty} \phi(x, z)e^{i\zeta(z \mp a)}dz$$

$$\Phi_{\mp}(x, \zeta) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \phi(x, z)e^{i\zeta(z \pm a)}dz$$

In view of the radiation condition, we can verify that $\Phi(x, \zeta)$ and $\Phi_{\pm}(x, \zeta)$ are regular in $|\tau| < k_2 \cos \theta_0$.
and $\tau \geq k_2 \cos \theta_0$, respectively, whereas $\Phi_1(x, z)$ is an entire function.

In view of the boundary conditions for tangential electromagnetic fields, we see that

$$E_t^x(\pm b, z) = 0, \quad |z| > a$$

(7)

$$E_t^x(\pm b + 0, z) = E_t^c(\pm b - 0, z), \quad |z| < a$$

(8)

$$H_t^x(\pm b + 0, z) = H_t^c(\pm b - 0, z), \quad |z| < a$$

(9)

where the superscript $t$ implies the total field. The tangential component of the total electric field is continuous across $x = \pm b$ for $-\infty < z < \infty$. In contrast, the tangential component of the total magnetic field is continuous across $x = \pm b$ only for $|z| < a$.

Taking the Fourier transform of the two-dimensional Helmholtz equation and solving the resulting transformed wave equations, we derive that

$$\Phi(x, z) = A(z)e^{-i(x-b)}, \quad x > b$$

$$= A(z)\sinh[\gamma(x + b)]/\sinh(2\gamma b)$$

$$- B(z)\sinh[\gamma(x - b)]/\sinh(2\gamma b), \quad |x| < b$$

$$= B(z)e^{i(x+b)}, \quad x < -b$$

(10)

where $\gamma = (x^2 - k^2)^{1/2}$ with $\Re \gamma > 0$, and $A(z)$ and $B(z)$ are unknown spectral functions.

Given the boundary conditions in the Fourier transform domain, we obtain that

$$A(z) + B(z) = S_1(z)$$

(11)

$$A(z) - B(z) = D_1(z)$$

(12)

$$e^{-i\tau a}J_-(b, x) + e^{i\tau a}J_+(b, x)$$

$$- (A_1e^{i\tau a} - A_2e^{-i\tau a})/(x - k \cos \theta_0)$$

$$= -\gamma A(z)[\cos(2\gamma b)/\sin(2\gamma b) + 1]$$

$$+ \gamma B(z)/\sin(2\gamma b)$$

(13)

$$\begin{align*}
\Phi_1(x, z) &= \Phi_1'(x, z)h, \\
S_1(x) &= \Phi_1(b, x) + \Phi_1(-b, x) \\
D_1(x) &= \Phi_1(b, x) - \Phi_1(-b, x) \\
J_\pm(\pm b, x) &= \Phi_\pm'(\pm b + 0, x) - \Phi_\pm'(\pm b - 0, x) \\
A_{12} &= -2k \sin \theta_0 e^{-ikb \sin \theta_0} e^{-i\tau a \cos \theta_0}/(2\pi)^{1/2}
\end{align*}$$

(14)

(15)

(16)

(17)

(18)

In (17), the prime denotes differentiation with respect to $x$.

Carrying out some manipulations with the aid of boundary conditions, we arrive at

$$-S_1(x)/L(x) = e^{i\tau a}U_+(x) + e^{-i\tau a}U_-(x)$$

(19)

$$-D_1(x)/N(x) = e^{i\tau a}V_+(x) + e^{-i\tau a}V_-(x)$$

(20)

where

$$\begin{align*}
U_+(x) &= J_+(x) + A_{12}/(x - k \cos \theta_0) \\
V_+(x) &= J_+(x) + A_{12}/(x - k \cos \theta_0) \\
L(x) &= e^{-i\gamma \cosh(\gamma b)}/\gamma \\
N(x) &= e^{-i\gamma \sinh(\gamma b)}/\gamma
\end{align*}$$

(21)

(22)

(23)

(24)

with

$$J_\pm(x) = J_\pm(b, x) + J_\pm(-b, x)$$

(25)

$$J_{12}(x) = J_+(b, x) - J_+(b, -x)$$

(26)

In the previously mentioned notations, the subscripts + and − imply that the functions are regular in the upper ($\tau > -k_2 \cos \theta_0$) and the lower ($\tau < k_2 \cos \theta_0$) half plane, respectively, whereas the subscript ($) implies that the functions are regular in $\tau > -k_2 \cos \theta_0$, except for a simple pole at $x = k \cos \theta_0$. Thus, (19) and (20) are the desired simultaneous Wiener–Hopf equations.

3. Solution of the Wiener–Hopf Equations

The kernel function defined by (23) and (24) can be factorized as in [9]

$$L(x) = L_+(x)L_-(x)$$

(27)
\[N(x) = N_+(x)N_-(x)\]  
(28)

where \(L_\pm(x)\) and \(N_\pm(x)\) are split functions defined by

\[
L_\pm(x) = [\cos(kb)]^{1/2} e^{i\pi/b} (k \pm x)^{-1/2} \cdot \exp \left( \pm i\pi b \right) \{ 1 - C + \ln[\pi/(2kb)] + i\pi/2 \} 
\cdot \exp \left( i\pi b \ln[(\pm x - \gamma)/k] \right) 
\cdot \prod_{n=1}^{\infty} \left[ 1 \pm \frac{x}{(i\pi n)} \right] e^{\pm 2\pi b \left( \pi n^{\pi} \right)}
\]  
(29)

\[
N_\pm(x) = [\sin(kb)]^{1/2} \cdot \exp \left( \pm i\pi b \right) \{ 1 - C + \ln[\pi/(2kb)] + i\pi/2 \} 
\cdot \exp \left( i\pi b \ln[(\pm x - \gamma)/k] \right) 
\cdot \prod_{n=1}^{\infty} \left[ 1 \pm \frac{x}{(i\pi n)} \right] e^{\pm 2\pi b \left( \pi n^{\pi} \right)}
\]  
(30)

with \(C = 0.57721566\) being the Euler constant, where

\[
\gamma_n = -ik, \quad n = 0, \\
= \left\{ \left[ \frac{n\pi}{(2b)} \right]^2 - k^2 \right\}^{1/2}, \quad n \geq 1
\]  
(31)

Carrying out the factorization and decomposition procedures for (19) and making some arrangements, we arrive at

\[
U_{\pm}^{\text{in,d}}(x) = \left[ - \frac{A_1 L_+ (k \cos \theta_0)}{\alpha + k \cos \theta_0} + \frac{A_2 L_- (k \cos \theta_0)}{\alpha - k \cos \theta_0} \right] \pm \frac{1}{\pi i} \int_k^{k+2\pi} F_+ (\beta) \left[ \frac{U_{\pm}^{\text{in,d}}(\beta)}{(\beta - k)^{1/2} (\beta + k)} \right] e^{2i\theta_d} d\beta \right]/L_+ (x)
\]  
(32)

where

\[
F_+ (\beta) = \cosh^2 (\gamma b) \left[ L_+ (\beta) (\beta + k)^{1/2} \right]
\]  
(33)

Therefore, (32) provides an exact solution of (19), but it is formal because it involves infinite branch-cut integrals with unknown integrals. We need to derive explicit approximate expressions of (32) for the solution to be complete.

By using the method of asymptotic solutions developed by \([7, 8]\) and making some manipulations, as was done in Part I \([6]\), we finally obtain that

\[
U_{\pm}^{\text{in,d}}(x) \sim \left[ \frac{A_1 L_+ (k \cos \theta_0)}{\alpha + k \cos \theta_0} + \frac{A_2 L_- (k \cos \theta_0)}{\alpha - k \cos \theta_0} \right] \pm \frac{1}{\pi i} \int_k^{k+2\pi} F_+ (\beta) \left[ \frac{U_{\pm}^{\text{in,d}}(\beta)}{(\beta - k)^{1/2} (\beta + k)} \right] e^{2i\theta_d} d\beta \right]/L_+ (x)
\]  
(34)

where

\[
f_{\text{in,d}}^{\text{u,v}} = \frac{1}{n!} \frac{d^n}{d\beta^n} \left[ J_n (\beta) \pm J_n^- (\beta) \right] \bigg|_{\beta=k}
\]  
(35)

\[
\eta_{1,2} (x) = \frac{\xi_{00}^+ (x) - \xi_{00}^- (x) \pm k \cos \theta_0}{\alpha + k \cos \theta_0}
\]  
(36)

\[
\xi_{\text{in}}^+ (x) = \frac{e^{i(2ka-3\pi/4)n}}{\pi (2a)^{n-1/2}} \Gamma_n^+ \left[ 1/2 + n, -2i(x+k)a \right]
\]  
(37)

\[
\Gamma_n^+ (u, v) = \int_0^\infty F_n^+ [k + it/(2a)] (t + v) \cdot \tau^{n-1} e^{-\tau} d\tau
\]  
(38)

In (34), the unknowns \(f_n^{\text{u,v}}\) are determined by solving the appropriate matrix equations. The special function defined by (38) is the generalized gamma function introduced by the authors \([8, 10]\) and rigorously accounts for the multiple-edge diffraction.

Thus, (34) provides high-frequency asymptotic solutions of the Wiener–Hopf equation, (19), and is valid for the waveguide slit width \(2a\) being comparable to the incident wavelength or larger. Further Wiener–Hopf equation given by (20) can be solved in the same manner as mentioned previously, resulting in similar asymptotic solutions; hence, the details are omitted.

### 4. Scattered Field

The inverse Fourier transform of the scattered field is defined as follows:

\[
\Phi(x, z) = (2\pi)^{-1/2} \int_{-\infty+i\epsilon}^{\infty-i\epsilon} \Phi(x, z) e^{-izd} dx,
\]

\[-k_2 < c < k_2 \cos \theta_0 \]  
(39)

First, we consider the region \(|x| > b\) and derive the scattered far-field expression. We can easily show from (10), (19), and (20) that the scattered field for this region is expressed as

\[
\Phi(x, z) = \tilde{\Phi}(x) e^{izd}
\]  
(40)

for \(x \gg b\), where

\[
\tilde{\Phi}(x) = \cosh \gamma b \left[ e^{izd} U_{\text{in}}(x) + e^{-izd} U_{\text{in}}(x) \right] / (2\gamma)
\]

\[
\mp \sinh \gamma b \left[ e^{izd} V_{\text{in}}(x) + e^{-izd} V_{\text{in}}(x) \right] / (2\gamma), \quad x \gg b
\]  
(41)

In (41), \(U_{\text{in}}(x), U_{\text{in}}(x)\) and \(V_{\text{in}}(x), V_{\text{in}}(x)\) are the solutions to the Wiener–Hopf equations (19) and (20), respectively. Introducing the cylindrical coordinates \(x = \rho \sin \theta, \quad z = \rho \cos \theta \ (-\pi < \theta < \pi)\) and applying the saddle point method, the scattered far field is derived as

\[
\phi(\rho, \theta) \sim \tilde{\Phi}(-k \cos \theta, \theta) \sin \theta e^{(k \rho - \pi/4)/(k \rho)}
\]  
(42)

for \(k \rho \to \infty\).

Next, we consider the region \(|x| < b\) and derive the scattered field inside the waveguides. For this region, we
substitute the field expression for $x_{jj}$, $b$ in (10) into (39) and take into account the solutions of the Wiener–Hopf equations. By computing the residues at an infinite number of simple poles of the integrands, we obtain that

$$x(z) = \sum_{n=1}^{\infty} \left[ e^{\pm \gamma_n} (-1)^n n \pi^3 / (2^{3/2} b^2 \gamma_n) \right] \cdot \left[ \Phi_1(b, \pm i \gamma_n) \sin[n\pi(x \pm b)/(2b)] \right]$$

$$- \Phi_1(-b, \pm i \gamma_n) \sin[n\pi(x \mp b)/(2b)] \right]$$

This completes derivation of the scattered field in real space.

5. Numerical Results and Discussion

We present numerical results on the far-field intensity and discuss scattering characteristics of the waveguide slit. For convenience, we define the normalized far-field intensity as

$$|\phi(\rho, \theta)|[\text{dB}] = 20 \log_{10} \left[ \frac{\lim_{\rho \to \infty} (k \rho)^{1/2} \phi(\rho, \theta)}{\max_{\rho \in [0, \infty]} (k \rho)^{1/2} \phi(\rho, \theta)} \right]$$

In numerical computation, we have chosen the same parameters as in the $H$-polarized case [6].

Figure 2 shows the normalized far-field intensity as a function of observation angle $\theta$, where the slit width and the waveguide spacing are chosen as $ka = 10$, 30 and $kb = 1$, 4, 7, 10, respectively, and the incidence angle has been fixed as $\theta_0 = 60^\circ$. Basic features observed for $E$ polarization in this figure are similar to those for $H$ polarization. In particular, for all chosen values of $ka$ and $kb$, the far-field intensity shows noticeable peaks along the reflection boundary at $\theta = 120^\circ$, as expected. We also see that for negative $\theta$, the peak location gradually shifts from the incident shadow boundary at $-120^\circ$, with an increase of $kb$.

Figure 3 shows comparison of the results for $kb = 0$ and $kb = 0$ (slit in an infinite plane). We observe that the two curves become very close to each other, as expected.

Finally, we compare the characteristics between $H$ and $E$ polarizations. Figure 4 shows the far-field intensity for $\theta_0 = 60^\circ$, $ka = 30$, $kb = 1$, where the solid (black) and dashed (red) curves denote $E$ and $H$ polarizations, respectively. From the figure, we note that the intensity for positive $\theta$ shows close features for both polarizations except around $\theta = 0^\circ$, 180°. For negative $\theta$, however, there are some differences between $E$ and $H$ polarizations. In particular, the intensity for both polarizations shows close features in the neighborhood of the incident direction, but except the main lobe.
direction, differences are clearly seen between two polarizations.

6. Conclusion

In this two-part article, we have rigorously analyzed the plane-wave diffraction by a slit formed by two semi-infinite parallel-plate waveguides for both $H$ polarization (Part I [6]) and $E$ polarization (Part II, this article) using the Wiener–Hopf technique. The geometry considered here is of the modified Wiener–Hopf geometry of the first kind; hence, we have applied the method of solution developed in our previous article [7] to obtain a high-frequency asymptotic solution for this waveguide slit problem. Note that the geometry considered in this article belongs to a class of classical problems, but the final results obtained here are totally new. In particular, the high-frequency solutions obtained here rigorously consider the multiple-edge diffraction effects; hence, the results are even valid for the slit width comparable to the incident wavelength.

7. References