Shortcut to Mie Scattering Through Complex Space

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Abstract – The scattering and extinction characteristics of a dielectric sphere by a plane wave can be analyzed using the Mie-scattering principles. The approach makes use of sums in which coefficients depend on the permittivity of the sphere, the frequency, and the size of the sphere through spherical Bessel and Hankel functions. To approximate the scattering results without a need to resort to this exact approach, in this article we show a way to predict approximate results using a single electric dipole component. The idea is to place the dipole into a complex point in space, applying the idea proposed by I. V. Lindell in 1987. Comparing the scattering patterns and forward-to-backscattering cross sections, we show that the complex-space approach generates reasonable approximations for spheres whose size parameter (circumference divided by the wavelength) can reach values over 1.

1. Introduction

The effect of electric and magnetic fields on material media leads to charges and currents that, in the linear response domain, are proportional to the excitation. For the mathematical analysis of the interaction between waves and matter, these effects are usually condensed into so-called constitutive relations between the field vectors and flux densities. These relations are not always straightforward. The richness and complexity of possible electromagnetic and optical responses for materials can include anisotropic, chiral, nonreciprocal, gyrotropic, magnetoelectric, and several other effects, as witnessed by the development of metamaterials during the present century [1].

When the interaction of electromagnetic waves with obstacles is analyzed, these constitutive parameters play an important role. However, when particles and objects composed of homogeneous materials are exposed to electromagnetic waves, their shape, size, and other geometric parameters also play a role in their global response. Indeed, even very simple homogeneous scatterers may display optically fascinating and complicated responses, as is witnessed by the full complexity of a rainbow, which is a result of the interaction of natural light with simple spherical water droplets. In this article, the scattering problem of electromagnetic waves by particles with a simple dielectric response is treated from a novel perspective.

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Admittedly, the scattering by homogeneous objects is a profoundly studied problem that has a history of one and a half centuries, all the way from publications by Alfred Clebsch (1861) [2], John William Strutt (later Lord Rayleigh, 1871) [3], Ludvig Valentin Lorenz (1890) [4], and Gustav Mie (1908) [5]. The Mie-scattering machinery allows one to compute the scattering, absorption, and extinction of penetrable spheres to arbitrary precision. However, these computations require evaluation of sums that in the exact form have an infinite number of terms [6]. These expansions start to converge after a given number of terms, depending on the size of the scattering sphere. Nevertheless, the terms (given in terms of the so-called Mie coefficients) are functions of spherical Bessel and Hankel functions and hence require computational codes and numerical power in order to evaluate the final scattering parameters.

In the following, a shortcut is suggested to compute certain scattering properties of small and moderate-sized dielectric spheres. Revisiting the decades-old complex-space multipole expansion theory by Lindell [7], the low-frequency Rayleigh scattering is generalized to cover higher frequencies than purely those in the quasi-static regime, still retaining the simplicity of dipole radiation and hence avoiding spherical Bessel functions and Mie coefficient expansions.

2. Scattering Patterns of Dielectric Spheres

If a dielectric object is small in terms of the wavelength, it scatters electromagnetic waves like an electric dipole. The scattered power of this so-called Rayleigh scattering is proportional to the fourth power of the frequency f. However, not only the size but also the permittivity of the scatterer affect the response. Consider a dielectric isotropic sphere with relative permittivity ε onto which an x-polarized, $+z$-propagating plane wave is incident, as shown in Figure 1. The size parameter of the sphere is $x = ka = 2\pi a/\lambda = 2\pi af/c$, where c is the speed of light.

In Figure 2, effects of the size and permittivity on the directional power scattering pattern can be seen. When the size is small ($x = 0.1$), the scattering pattern is rotationally symmetric and has the $\sin^2 \theta_x$ dependence, where θ_x is the angle between the scattering direction and the electric field (x-axis). However, when x increases, the pattern becomes more directive, and it is also strongly affected by the relative permittivity ε. In each case, the maximum scattering is into the forward direction. The patterns have been computed with a full Mie-scattering code.

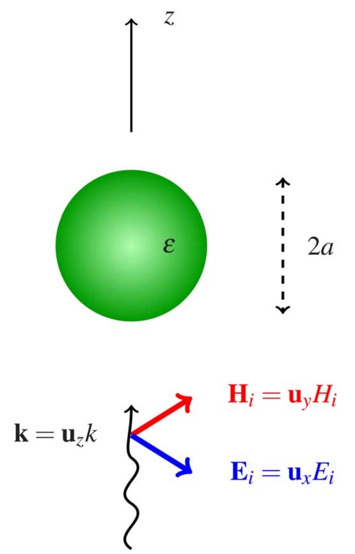

Figure 1. Geometry of the problem: scattering of a plane wave by a dielectric sphere. The incident wave propagates into the $+z$-direction, the electric field $\mathbf{E}_i$ is in the x-direction, and the magnetic field $\mathbf{H}_i$ in the y-direction.

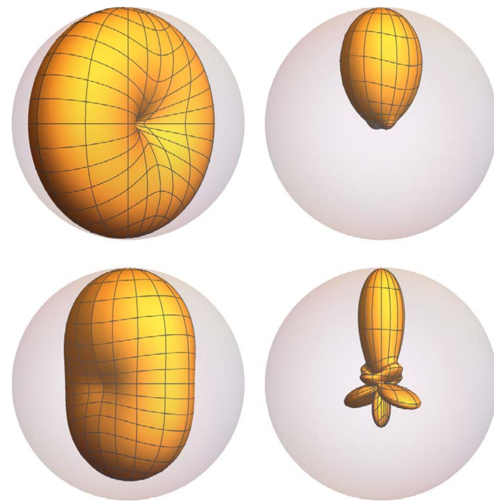

Figure 2. Power scattering patterns by a dielectric sphere illuminated by a plane wave propagating into the positive z-direction (upward in the figure). The polarization of the incident wave is as shown in Figure 1. The parameters of the spheres (relative permittivity ε and size parameter $x = 2\pi a/\lambda$) are the following: $\varepsilon = 3$, $x = 0.1$ (upper left); $\varepsilon = 3$, $x = 2$ (upper right); $\varepsilon = 10$, $x = 1$ (lower left); and $\varepsilon = 10$, $x = 3$ (lower right). The gray sphere denotes the magnitude of maximum radiation in each case.

3. Approximation of Dielectric Scattering by a Dipole in Complex Source Point

As Figure 2 shows, the scattering pattern deviates strongly from the dipole radiation when the electrical size of the sphere increases due to the higher-order electric and magnetic multipole terms. However, as has been shown by Lindell [7], by locating multipoles in complex space, the dominance of the first terms can be enhanced. Already in the 1970s, complex space point sources were introduced in electromagnetics to model and analyze the propagation and reflection of Gaussian beams [8–10]. Let us apply this finding and shift the electric dipole induced in the sphere into a position with the real part being the center of the sphere but having a complex position vector [7]

$$\mathbf{r}' = -\mathrm{j}b\mathbf{u}_z = -\mathrm{j}\,\frac{(\varepsilon+2)(\varepsilon+4)}{15(2\varepsilon+3)}ka^2\mathbf{u}_z \qquad (1)$$

Due to the fact that the phase term from a source point $\mathbf{r}'$ to the field point $\mathbf{r}$ is $\exp(-\mathrm{j}k(\mathbf{r}-\mathbf{r}'))$, there will be an additional exponential multiplying the dipolar radiation (here we follow the time-harmonic convention $\exp(\mathrm{j}\omega t)$). The normalized scattered power pattern becomes

$$S(\theta,\phi) = \left(\sin^2\varphi + \cos^2\varphi\cos^2\theta\right)\mathrm{e}^{2kb(\cos\theta-1)} \qquad (2)$$

where the angles are those of the ordinary spherical system (r,θ,φ). It is easy to see that (2) distills down to the real-space-located dipole pattern $\sin^2\theta_x$ when b becomes zero.

The ratio between forward and backward scattering cross sections of the complex dipole is

$$\frac{S(0,\varphi)}{S(\pi,\varphi)} = \mathrm{e}^{4kb} \qquad (3)$$

This can be compared with the exact solution from the Mie theory. The monostatic radar cross section normalized by the geometrical cross section of the sphere is called the backscattering efficiency, and it reads, using the Mie coefficients [11],

$$Q_\mathrm{b} = \frac{1}{x^2}\left|\sum_{n=1}^{\infty}(2n+1)(-1)^n(a_n-b_n)\right|^2 \qquad (4)$$

Likewise, the forward scattering efficiency can also be computed:

$$Q_f = \frac{1}{x^2}\left|\sum_{n=1}^{\infty}(2n+1)(a_n+b_n)\right|^2 \qquad (5)$$

where the electric and magnetic Mie coefficients are

$$a_n = \frac{\sqrt{\varepsilon}\,\psi_n(\sqrt{\varepsilon}x)\psi'_n(x) - \psi_n(x)\psi'_n(\sqrt{\varepsilon}x)}{\sqrt{\varepsilon}\,\psi_n(\sqrt{\varepsilon}x)\xi'_n(x) - \xi_n(x)\psi'_n(\sqrt{\varepsilon}x)}$$
$$b_n = \frac{\psi_n(\sqrt{\varepsilon}x)\psi'_n(x) - \sqrt{\varepsilon}\,\psi_n(x)\psi'_n(\sqrt{\varepsilon}x)}{\psi_n(\sqrt{\varepsilon}x)\xi'_n(x) - \sqrt{\varepsilon}\,\xi_n(x)\psi'_n(\sqrt{\varepsilon}x)} \qquad (6)$$

The Riccati–Bessel functions ψ_n, ξ_n are as follows:

$$\psi_n(\rho) = \rho\, j_n(\rho), \qquad \xi_n(\rho) = \rho\, h_n^{(2)}(\rho) \qquad (7)$$

where j_n and h_n are the usual spherical Bessel and Hankel functions of order n.

Figure 3 shows the front-to-back ratio $Q_\mathrm{f}/Q_\mathrm{b}$, both the exact one and the complex-dipole approximation. Remembering that an electric dipole in real space would

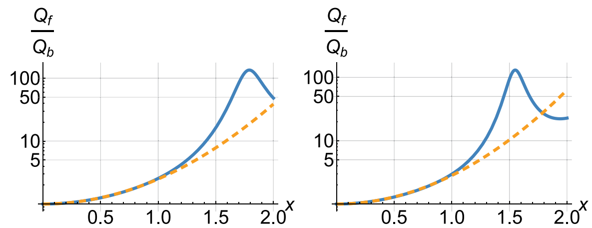

Figure 3. Front-to-back scattering ratio for the exact Mie solution (blue solid line) and approximation with complex-point dipole (dashed orange) for a sphere as function of the size parameter x for two permittivities. Left: $\varepsilon = 2$; right: $\varepsilon = 3$.

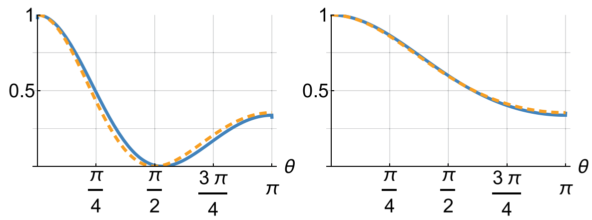

Figure 4. Exact Mie solution (blue solid line) and the approximation with complex-point dipole (dashed orange) for a sphere with relative permittivity $\varepsilon = 3$ and size parameter $x = 1$. E-plane (left) and H-plane (right).

predict the value $Q_f/Q_b = 1$, independently of x, we can conclude that positioning the dipole into complex space is a remarkable improvement, as this ratio can be approximated into size parameter values even exceeding $x = 1$.

The scattering patterns are compared in Figures 4 and 5. The relative scattered power is shown as a function of the scattering angle θ ($\theta = 0$ shows the forward scattering and $\theta = \pi$ the backscattering direction) in the two planes, $z - x$ (E-plane) and $z - y$ (H-plane), both for a sphere with relative permittivity $\varepsilon = 3$. The exact pattern is very well approximated by the complex dipole for the size parameter $x = 1$ (Figure 4). However, as the size increases ($x = 3$), only the main forward lobe is predicted reasonably, and the approximation fails to provide the details of side and backscattering (Figure 5).

The improvement caused by the shift of the dipole into complex space on the scattering prediction can be appreciated from the full three-dimensional scattering

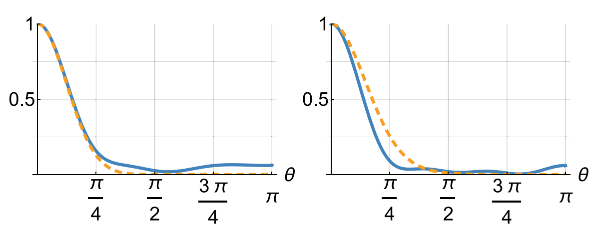

Figure 5. Exact Mie solution (blue solid line) and the approximation with complex-point dipole (dashed orange) for a sphere with relative permittivity $\varepsilon = 3$ and size parameter $x = 3$. E-plane (left) and H-plane (right).

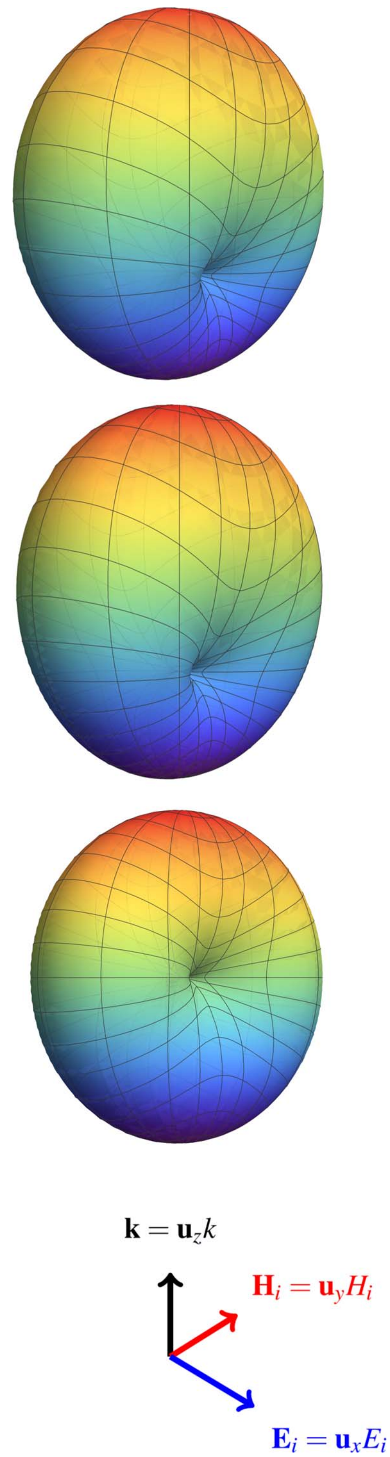

Figure 6. Comparison of dipole approximations against the exact scattering pattern for a dielectric sphere with relative permittivity $\varepsilon = 2$ and size parameter $x = 1$. Exact Mie-scattering pattern (top), complex source-point dipole scattering (middle), and scattering by an ordinary dipole in real space (bottom).

patterns, shown in Figure 6. As before, the incident wave propagates along the $+z$-axis, and it is linearly polarized with the electric field in the x-direction. Then a pure electric dipole, located in real space, radiates the rotationally symmetric pattern with equal forward and backscattering (lowest radiation pattern of the three in the figure). However, the shift into complex space makes the pattern asymmetric between the forward half-space ($0 \leq \theta \leq \pi/2$) and the backward half-space ($\pi/2 \leq \theta \leq \pi$). Indeed, a visual comparison between the exact pattern (top of the figure) and the complex-space approximation (middle pattern) does not reveal any notable differences.

4. Conclusion

The message of this article was that, while studying the scattering by a finite-sized dielectric sphere, the simple electric dipole approximation can be considerably made more exact by moving the location of the dipole into a complex space point from the center into the direction of the propagating wave. The exact forward-to-backscattering cross sections can be predicted with good accuracy up to size parameter values exceeding unity. Of course, for larger spheres where many electric and magnetic multipoles contribute to the scattered fields, the approximation fails. For example, the behavior of the forward-to-backscattering ratio increases exponentially with increasing size for the complex-space dipole, as shown in (3). This is the case also in the exact Mie solution, as seen, for example, in Figure 3. However, when the size and permittivity of the sphere increase, this ratio starts to oscillate and behave irregularly. Nevertheless, due to its simplicity, the complex-space dipole approximation may turn useful in the analysis of scattering by small and moderate-sized spheres and applications to material modeling and the computation of the effective permittivity of heterogeneous media.

5. References

1. F. Capolino (ed.), *Metamaterials Handbook*, CRC Press, Boca Raton, FL, 2009.
2. A. Clebsch, “Ueber die Reflexion an einer Kugelflache,” *Journal für Mathematik*, **61**, 3, 1863, pp. 195-262.
3. J. W. Strutt, “XV. On the Light From the Sky, Its Polarization and Colour,: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, **41**, 271, 1871, pp. 107-120.
4. Ludvig V. Lorenz, “Lysbevægelser i og uden for en af plane Lysbølger belyst Kugle,” *Kongelige Danske Videnskabernes Selskabs Skrifter*, **6**, 1890, pp. 2-62.
5. Gustav Mie, “Beiträge zur Optik trüber Medien, speziell kolloidaler Metallösungen,” *Annalen der Physik*, **25**, 1908, pp. 377-445.
6. Craig R. Bohren and Donald R. Huffman, *Absorption and Scattering of Light by Small Particles*, Wiley, New York, 1983.
7. I. V. Lindell, “Complex Space Multipole Expansion Theory With Application to Scattering From Dielectric Bodies,” *IEEE Transactions on Antennas and Propagation*, **35**, 6, 1987, pp. 683-689.
8. G. A. Deschamps, “Gaussian Beam as a Bundle of Complex Rays,” *Electronics Letters*, **23**, 1971, pp. 684-685.
9. S. Y. Shin and L. B. Felsen, “Gaussian Beam Modes by Multipoles With Complex Source Points,” *Journal of the Optical Society of America*, **67**, 5, 1977, pp. 699-700
10. A. L. Cullen and P. K. Yu, “Complex Source-Point Theory of the Electromagnetic Open Resonator,” *Proceedings of the Royal Society A*, **366**, June 1979, pp. 155-171.
11. Ari Sihvola, Henrik Wallén, Pasi Ylä-Oijala, and Rashda Parveen, “Enhanced Backscattering for Dielectrically Active Scatterers,” *URSI Radio Science Letters*, **3**, 201, doi:10.46620/21-0020.