Rounding the Corners of Scatterers: A Perturbation Analysis of Far-Field Changes

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Abstract – This article presents an improved analysis of the changes to surface and far-field quantities induced by rounding the sharp corners of a two-dimensional scatterer. A prototypical scatterer with one sharp corner is considered. It uses a perturbation analysis of the integral equation formulation for this scatterer on the basis of the double-layer potential. This approach considerably simplifies previous treatment by the authors, generalizes it to an arbitrary interior angle at the corner, and improves the transparency of the main result.

1. Introduction

There is extensive literature on scattering and diffraction from sharp-cornered objects, as well as those with smooth boundaries. However, there are few studies of the transition from one to the other as the radius of curvature of the rounded corner points tends to zero. A survey of relevant studies is in [1, 2]. Numerical studies [3, 4] of single-cornered structures with a range of interior angles 2Ω showed that the nondimensionalized maximum difference $\sqrt{k}||u_0^\infty - u_\rho^\infty||_\infty$ in the far field u_0^∞ of the sharp-cornered structure and that of its rounded counterpart u_ρ^∞ under illumination by a time-harmonic E-polarized plane wave of wavenumber k, is bounded by $C(\theta_0)(k\rho)^{2/v}$ as $k\rho \to 0$, where the constant $C(\theta_0)$ depends upon the angle of incidence θ_0, the (minimum) radius ρ of curvature at the rounded corner, and $v = (2\pi - 2\Omega)/\pi$. The analytic studies [1, 3] established this result for a right-angled corner ($2\Omega = \pi/2$) by considering the difference of the integral equations obeyed by the surface distributions and estimating the difference in the corresponding surface distributions. The purpose of this article is to present a simpler and more transparent approach, valid for all angles $0 < 2\Omega < \pi$, by considering the rounded structure and its surface distribution to be a perturbation of the corresponding sharp-cornered scatterer and its distribution.

2. Formulation

Consider an infinitely long, perfectly electrically conducting cylinder of uniform cross-section D, axis parallel to the z-axis and illuminated by a time-harmonic E-polarized plane wave $u^{\mathrm{inc}}(\mathbf{x}) = e^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$ of wavenumber k, propagating with direction $\mathbf{d} = (\cos\theta_0, \sin\theta_0)$ parallel to the x-y plane. The time-harmonic factor $e^{-\mathrm{i}\omega t}$ is suppressed throughout. The incident field induces a field u^{sc} scattered by the obstacle so that the total field satisfies the Helmholtz equation exterior to D and vanishes on the closed boundary ∂D; the scattered field u^{sc} obeys the Sommerfeld radiation condition and the finiteness of energy condition in the vicinity of the corner.

Let ∂D be parametrized by

$$\mathbf{x} = \mathbf{x}(t) = (x_1(t), x_2(t)), t \in [-\pi, \pi] \tag{1}$$

We will examine a scatterer ∂D_0 that can be regarded as prototypical, namely, the lemniscate with interior corner angle 2Ω, parametrized by

$$\mathbf{x} = \mathbf{x}_0(t) = a(2\sin(|t|/2), -\tan\Omega\sin(t)), [-\pi, \pi] \tag{2}$$

where a is a parameter equal to one length unit. Consider a family of curves ∂D_ε in which the corners have been rounded, having representation

$$\begin{aligned}\mathbf{x} &= \mathbf{x}_\varepsilon(t) \\ &= a\left(2\sqrt{\varepsilon^2 + (1-\varepsilon^2)\sin^2(t/2)}, -\tan\Omega\sin(t)\right)\end{aligned} \tag{3}$$

where $t \in [-\pi, \pi]$ and parametrized by the quantity ε ($0 < \varepsilon < 1$). Denote the radius of curvature at the point $\mathbf{x}_\varepsilon(t)$ by $\rho(t)$; its value $\rho = \rho(0)$ at the corner point $\mathbf{x}_\varepsilon(0)$ satisfies $\rho = 2\varepsilon\tan^2\Omega + O(\varepsilon^3)$, as $\varepsilon \to 0$.

As explained in [5], the solution to the exterior Dirichlet problem is based on representing the scattered field as a double-layer potential of a continuous surface density $\phi = \phi(\mathbf{y})$ that is defined on ∂D:

$$u^{\mathrm{sc}}(\mathbf{x}) = \frac{1}{2}K\phi(\mathbf{x}) = \int_{\partial D}\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{y})}\phi(\mathbf{y})ds(\mathbf{y}) \tag{4}$$

Here, G denotes the two-dimensional free-space Green's function $G(\mathbf{x},\mathbf{y}) = (i/4)H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|)$, and $\mathbf{n} = \mathbf{n}(\mathbf{y})$ denotes the outward unit normal at the point $\mathbf{y}$. The surface distribution is then determined by the integral equation on ∂D,

$$\phi(\mathbf{x}) + K\phi(\mathbf{x}) = -2u^{\mathrm{inc}}(\mathbf{x}) \tag{5}$$

Uniqueness and solubility are discussed in [5]. Henceforth, we suppose that k is a wavenumber at which there is a unique solution to this integral equation; once the solution ϕ has been found, the far field is readily calculated.

3. Perturbation Analysis

Let K_0 and K_ε denote the double-layer operators corresponding to the surface ∂D_0 and its rounded

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counterpart ∂D_ε, respectively; let $\phi_0 = \phi_0\,(\mathbf{x}_0)$ and $\phi_\varepsilon = \phi_\varepsilon\,(\mathbf{x}_\varepsilon)$ be the corresponding surface densities.

Introduce the perturbation operator $\Delta K = K_\varepsilon - K_0$ and the perturbation density $\Delta\phi = \phi_\varepsilon - \phi_0$; also, let $g_0\,(\mathbf{x}_0) = -2u^{\mathrm{inc}}(\mathbf{x}_0)$, $g_\varepsilon\,(\mathbf{x}_\varepsilon) = -2u^{\mathrm{inc}}(\mathbf{x}_\varepsilon)$, so $\Delta g = g_\varepsilon - g_0$ measures the field perturbation at corresponding points on the surface obstacle and its rounded counterpart. Thus,

$$(I + K_0)\phi_0 = g_0 \tag{6}$$

and

$$(I + K_0 + \Delta K)(\phi_0 + \Delta\phi) = g_0 + \Delta g \tag{7}$$

Neglecting the *double-perturbation* term, $\Delta K\ (\Delta\phi)$ leads to the following approximate integral equation for the quantity $\Delta\phi$:

$$(I + K_0)(\Delta\phi) = \Delta g - \Delta K(\phi_0) \tag{8}$$

We now proceed as in [1] to analyze (8). Let I be a symmetrical interval of $[-\pi, \pi]$ and set $J = [-\pi, \pi] \setminus I$, with the following requirements on I: on the set J, the maximum difference between the parametrizations $\mathbf{x}_0$ and $\mathbf{x}_\varepsilon$ is negligibly small, as well as that between the derivatives $\mathbf{x}_0'$ and $\mathbf{x}_\varepsilon'$. The maximum value of $\Delta\phi$ on J is negligibly small, and the interval I is electrically small; that is, $k|I|$ is small, so small argument approximations may be used for the Hankel function $H_1^{(1)}$. In fact, the interval I may be taken to be $\left[-\varepsilon^{1/v}, \varepsilon^{1/v}\right]$.

This last requirement equivalently means that for points on the relevant surface parametrized by I, the Green's function and its normal derivative may be well approximated by the corresponding values of the static Green's function and its normal derivative. Thus, making the approximation $\Delta\phi\ (t) = 0$ for $t \in J$, and setting $\psi\ (t) = \Delta\phi\ (t)$, the explicit form of the term $K_0\ (\Delta\phi)\ (t)$ is [5]:

$$\frac{1}{\pi}\int_{-\varepsilon^{1/v}}^{\varepsilon^{1/v}} \frac{\mathbf{n}(\mathbf{x}_0(\tau)) \cdot (\mathbf{x}_0(t) - \mathbf{x}_0(\tau))}{|\mathbf{x}_0(t) - \mathbf{x}_0(\tau)|^2}\left|\mathbf{x}_0'(\tau)\right|\psi(\tau)d\tau \tag{9}$$

On the interval I, the scatterer ∂D_0 having corner at $t = 0$ is well approximated by $\mathbf{x}_0\,(t) = (|t|,\, t\tan\Omega)$, i.e., part of an infinite wedge of angle 2Ω. Inserting this in the integral, one obtains the bound

$$\|\psi\|\frac{1}{\pi}\int_0^{\varepsilon^{1/v}} \frac{t\sin 2\Omega}{t^2 - 2t\tau\cos 2\Omega + \tau^2}d\tau < \|\psi\|\left(1 - \frac{2\Omega}{\pi}\right) \tag{10}$$

having replaced the upper limit by ∞. The norm $\|\cdot\|$ used here is the maximum norm of the function, and for an operator, the norm $\|\cdot\|$ denotes that induced by this maximum norm. Because $0 < 2\Omega < \pi$, we have shown that $\|K_0\psi\| \le \left(1 - \frac{2\Omega}{\pi}\right)\|\psi\|$ so that the norm of the operator K_0 is less than unity: $\|K_0\| < 1$.

A similar argument is used to show that the norm of the operator K_ε is less than unity: $(\|K_\varepsilon\| < 1)$. The corresponding calculation for the operator is

$$\frac{1}{\pi}\int_{-\varepsilon^{1/v}}^{\varepsilon^{1/v}} \frac{\mathbf{n}(\mathbf{x}_\varepsilon(\tau)) \cdot (\mathbf{x}_\varepsilon(t) - \mathbf{x}_\varepsilon(\tau))}{|\mathbf{x}_\varepsilon(t) - \mathbf{x}_\varepsilon(\tau)|^2}\left|\mathbf{x}_0'(\tau)\right|\psi(\tau)d\tau \tag{11}$$

On the interval I, the rounded scatterer has the form $\mathbf{x}_\varepsilon(\tau) \sim \left(x_{\varepsilon,1}(\tau), -\tan\Omega\sin(\tau)\right)$ so that

$$\begin{aligned} \mathbf{x}_\varepsilon'(\tau) &\sim \left(x_{\varepsilon,1}'(\tau), -\tan\Omega\right), \\ \left|\mathbf{x}_\varepsilon'(\tau)\right|^2 &\sim \left(x_{\varepsilon,1}'(\tau)\right)^2 + \tan^2\Omega \end{aligned} \tag{12}$$

and

$$\mathbf{x}_\varepsilon''(\tau) \sim \left(x_{\varepsilon,1}''(\tau), 0\right) \tag{13}$$

By inserting, for the denominator, a first-order Taylor expansion, and for the numerator, the second-order Taylor approximation

$$\mathbf{x}_\varepsilon(t) = \mathbf{x}_\varepsilon(\tau) + (t - \tau)\mathbf{x}_\varepsilon'(\tau) + \frac{1}{2}(t - \tau)^2\mathbf{x}_\varepsilon''\left(\frac{t + \tau}{2}\right) \tag{14}$$

one deduces that $||K_\varepsilon|| < 1$. The details are in [1, 3].

We are now in a position to apply a norm analysis to the approximate integral equation (8) for the quantity $\Delta\phi$. Because $||K_0|| < 1$, $\left\|(I + K_0)^{-1}\right\| \le (1 - \|K_0\|)^{-1}$ is a bounded finite quantity. Thus,

$$\|\Delta\phi\| \le \left\|\,(I + K_0)^{-1}\right\|(\|\Delta g\| + (\|K_0\| + \|K_\varepsilon\|)\|\phi_0\|) \tag{15}$$

so

$$\|\Delta\phi\| \le (1 - \|K_0\|)^{-1}(\|\Delta g\| + 2\|\phi_0\|) \tag{16}$$

It remains to estimate the size of the terms $\Delta\phi$ and ϕ_0.

First, because $\Delta g = -2\left(u^{\mathrm{inc}}(\mathbf{x}_\varepsilon) - u^{\mathrm{inc}}(\mathbf{x}_0)\right)$,

$$\|\Delta g\| \le 2\left\|e^{ik\mathbf{x}_\varepsilon\cdot\mathbf{d}} - e^{ik\mathbf{x}_0\cdot\mathbf{d}}\right\| \le 2k\|\mathbf{x}_\varepsilon - \mathbf{x}_0\| \le 4k\varepsilon \tag{17}$$

Second, the estimate for ϕ_0 is derived from the total field in the vicinity of the tip of the infinite wedge of angle 2Ω; in terms of polar coordinates $(r,\ \theta)$ centered on the tip, it is given in [6] to be

$$\begin{aligned} u_0^{\mathrm{tot}}(r,\theta) = {} & \\ & \frac{4}{\Gamma(1/v)}\left(\frac{1}{2}kr\right)^{1/v} e^{-i\pi/v}\sin\left(\frac{\theta - \Omega}{v}\right)\sin\left(\frac{\theta_0 - \Omega}{v}\right) \\ & + O\left((kr)^{\min(2/v,1)}\right) \end{aligned} \tag{18}$$

It is readily deduced [1, 3] that

$$\phi_0(r) = A + B(kr)^{1/v} + O(kr) \tag{19}$$

as $kr \to 0$, where A and B are suitable constants; moreover, the constant A can be neglected because $\|\Delta K(A)\|$ is $O(k\varepsilon)$, as $k\varepsilon \to 0$. Inserting these estimates for Δg and ϕ_0 in (16), we deduce that

$$\|\Delta\phi\| \leq (1 - \|K_0\|)^{-1}\left(4k\varepsilon + 2B(k\varepsilon)^{1/v}\right) + O(k\varepsilon) \tag{20}$$

as $k\varepsilon \to 0$. Bearing in mind that $1/v < 1$, we conclude that $\|\Delta\phi)\| = O\left((k\varepsilon)^{1/v}\right)$, as $k\varepsilon \to 0$.

4. Conclusion

The perturbation $\Delta\phi$ in the surface density distribution has been determined by a perturbation analysis of the underlying integral equation formulation for the surface density distribution on the sharp-cornered scatterer under consideration. It may now be inserted in the expressions [5] for the far field u_0^∞ of the sharp-cornered structure and that of its rounded counterpart u_ε^∞ to obtain the corresponding perturbation in the far-field quantities. As shown in [1, 3], it readily follows that $\sqrt{k}\left\|u_0^\infty - u_\varepsilon^\infty\right\| = O\left((k\varepsilon)^{2/v}\right)$ as $k\varepsilon \to 0$. As the result is dependent upon the angle θ_0 of incidence of the illuminating plane wave and bearing in mind that the radius of curvature ρ at the corner point satisfies $\rho = 2\varepsilon \tan^2 \Omega + O(\varepsilon^3)$, as $\varepsilon \to 0$, we may restate the result in the form given in the introduction. The restriction $0 < 2\Omega < \pi$ on the corner angle precludes a structure with either a cusp at the corner or no genuine corner (i.e., a smooth surface).

The result is consistent with the numerical studies [3, 4] of single-cornered structures with a range of interior angles 2Ω, as well as structures with multiple corners (of the same interior angle). Those studies provided strong evidence that the maximum difference $\sqrt{k}\left\|u_0^\infty - u_\rho^\infty\right\|$ is not merely bounded by the term $O\left((k\rho)^{2/v}\right)$ but has the more precise estimate $C(\theta_0)(k\rho)^{2/v} + o\left((k\rho)^{2/v}\right)$, as $k\rho \to 0$, where the constant $C(\theta_0)$ depends upon the angle of incidence θ_0.

The advantage of the approach outlined in this article is that it considerably simplifies the previous treatment by the authors in [1, 3], it generalizes the main result to an arbitrary interior angle at the corner, and it improves the transparency of its derivation. It opens the way to consideration of the impact of differing types of rounding that might be applied to the corner.

5. References

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