Basic Differential Equations of Theoretical Physics After the Concept of Generalized Functions in the Sense of Distribution and the Double Current Sheet Problem of Van Bladel

To the memory of Jean Van Bladel

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Abstract – The double current sheet problem analyzed by Van Bladel some thirty years ago is reexamined in the light of the theory of distributions.

1. Distributions and the Derivatives in the Sense of Distribution

As is well known, almost all differential equations of theoretical physics had been established within the three centuries preceding the second quarter of the 20th century. They were admitted to be valid for differentiable (or piecewise differentiable) functions in the four-dimensional space. In just the middle of that century, Laurent Schwartz had published his theory of distributions [1], which introduced into the mathematical literature a new class of entities that were coined later on as the generalized functions. An appealing property of these functions was that without any restriction, they had derivatives of all orders. For example, for a vector-valued bounded and piecewise differentiable function $\boldsymbol{E}$, which has discontinuities on a regular surface S that moves arbitrarily, one has (see, e.g., [2])

$$\nabla\times\{\boldsymbol{E}\} = \{\nabla\times\boldsymbol{E}\} + [[\boldsymbol{n}\times\boldsymbol{E}]]\ |\nabla w|\ \delta(w) \tag{1a}$$

$$\nabla\bullet\{\boldsymbol{E}\} = \{\nabla\bullet\boldsymbol{E}\} + [[\boldsymbol{n}\bullet\boldsymbol{E}]]\ |\nabla w|\ \delta(w) \tag{1b}$$

$$\frac{\partial}{\partial t}\ \{\boldsymbol{E}\} = \left\{\frac{\partial}{\partial \mathrm{t}}\boldsymbol{E}\right\} - [[\boldsymbol{E}]]\ (\boldsymbol{n}\bullet\boldsymbol{v})\ |\nabla w|\ \delta(w) \tag{1c}$$

Here we assume that in an orthogonal coordinate system (u, v, w), the surface S is determined by $w = 0$ and $\boldsymbol{n}$ is the arbitrarily oriented unit normal vector to S, while $\boldsymbol{v}$ stands for the velocity of the surface S at the point $(u, v, 0)$ (see Figure 1); $\delta(w)$ taking place in (1a–1c) is the usual Dirac distribution. As to the quantities denoted by the curly braces and double brackets like $\{E\}$ and $[[E]]$, respectively, the first one stands for the expression of E outside S, which consists of a continuously differentiable function. It is called the regular part of E. The second one denotes the increment of E when one crosses S in the direction shown by $\boldsymbol{n}$, namely, $[[E]] = E(u,\ v,\ +0) - E(u,\ v,\ -0)$. The additional terms in (1a–1c), which involve the distribution $\delta(w)$, are called the singular parts of the corresponding quantities.

Apparition of the distribution concept had naturally caused the question:

> Are the already known differential equations of theoretical physics valid in the 4-dimensional space in the sense of distribution?

It goes without saying that the answer to this question is not naturally and imperatively affirmative because the generalization made in the sense of the distribution was in a rather particular direction chosen by Schwartz. For example, the multiplication of this kind of two generalized functions is not defined, which excludes the nonlinear equations such as the Navier–Stokes equations of the fluid mechanics. Moreover, the new physics results to be obtained with this assumption must also be compatible with the experiments. Therefore, an affirmative answer to the aforementioned question is a new postulate put on the equations in question.

Assumption of this postulate for the Maxwell equations in a nonhomogeneous four-dimensional space, composed of several simple domains having different constitutive parameters, yields very important and interesting issues (see, e.g., [2–7]). Now I would like to review some of these briefly.

2. Maxwell Equations in the Sense of Distribution

Now let us consider the Maxwell equations

$$\nabla\times\boldsymbol{H} - \partial\boldsymbol{D}/\partial t = \boldsymbol{J},\ \nabla\times\boldsymbol{E} + \partial\boldsymbol{B}/\partial t = 0 \tag{2a}$$

$$\nabla\bullet\boldsymbol{D} = \rho,\ \nabla\bullet\boldsymbol{B} = 0 \tag{2b}$$

which enforce also the continuity equation

$$\nabla\bullet\boldsymbol{J} + \partial\rho/\partial t = 0 \tag{2c}$$

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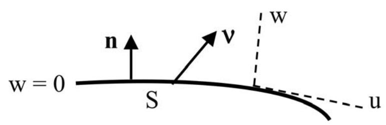

Figure 1. A moving discontinuity surface.

for the source densities. Assume that these equations are valid in the sense of distribution in a space divided into parts by a certain surface S as shown in Figure 1. Since some components of the field exhibit jump-type discontinuities on the interface S, the electric field $\boldsymbol{E}$, for example, has an expression of the following form near a small neighborhood of S [2, p. 52]:

$$\boldsymbol{E} = \{\boldsymbol{E}\} + \boldsymbol{E}_0\delta(w) + \boldsymbol{E}_1\delta\,'(w) + \ldots + \boldsymbol{E}_m\ \delta^{(m)}(w) \quad (3)$$

The coefficients $\boldsymbol{E}_j(u, v, 0, t)$ are all functions defined on the surface S, while m appearing on the last term is the order (in the sense of distribution) of $\boldsymbol{E}$. Similar expressions are also valid for the other components of the field as well as the densities of the source (i.e., for ρ and $\mathbf{J}$). Notice that the order m can be assumed to be the same for all components because we can prolong (3) with zero coefficients.

If we insert (3) and (1a–1c) into the Maxwell equations (2a–2c) and equate the terms of the same order taking place in the left- and right-hand sides, then we observe that the regular parts are reduced to the already known relations valid inside the simple domains separated by S (a trivial result), while the coefficients of $\delta^{(k)}(w)$ yield [2, p. 56]

$k = 0 \Rightarrow$ universal boundary conditions

$$\lambda[[\boldsymbol{n} \times \boldsymbol{H}]] + \lambda v_n[[\boldsymbol{D}]] = -\nabla \times \boldsymbol{H}_0 + \tfrac{\partial}{\partial t}\boldsymbol{D}_0 + \boldsymbol{J}_0 \quad (4a)$$

$$\lambda[[\boldsymbol{n} \times \boldsymbol{E}]] - \lambda v_n[[\boldsymbol{B}]] = -\nabla \times \boldsymbol{E}_0 - \tfrac{\partial}{\partial t}\boldsymbol{B}_0 \quad (4b)$$

$$\lambda[[\boldsymbol{n} \bullet \boldsymbol{D}]] = \rho_0 - \nabla \bullet \boldsymbol{D}_0 \quad (4c)$$

$$\lambda[[\boldsymbol{n} \bullet \boldsymbol{B}]] = -\nabla \bullet \boldsymbol{B}_0 \quad (4d)$$

$$\lambda[[\boldsymbol{n} \bullet \boldsymbol{J}]] - \lambda v_n[[\rho]] = -\tfrac{\partial}{\partial t}\rho_0 - \nabla \bullet \boldsymbol{J}_0 \quad (4e)$$

$k \geq 1 \Rightarrow$ compatibility conditions

$$\nabla \times \boldsymbol{H}_k + \lambda \boldsymbol{n} \times \boldsymbol{H}_{k-1} - \tfrac{\partial}{\partial t}\boldsymbol{D}_k + \lambda v_n \boldsymbol{D}_{k-1} = \boldsymbol{J}_k \quad (5a)$$

$$\nabla \times \boldsymbol{E}_k + \lambda \boldsymbol{n} \times \boldsymbol{E}_{k-1} + \tfrac{\partial}{\partial t}\boldsymbol{B}_k - \lambda v_n \boldsymbol{B}_{k-1} = 0 \quad (5b)$$

$$\nabla \bullet \boldsymbol{D}_k + \lambda \boldsymbol{n} \bullet \boldsymbol{D}_{k-1} = \rho_k \quad (5c)$$

$$\nabla \bullet \boldsymbol{B}_k + \lambda \boldsymbol{n} \bullet \boldsymbol{B}_{k-1} = 0 \quad (5d)$$

$$\nabla \bullet \boldsymbol{J}_k + \lambda \boldsymbol{n} \bullet \boldsymbol{J}_{k-1} + \tfrac{\partial}{\partial t}\rho_k - \lambda v_n \rho_{k-1} = 0 \quad (5e)$$

Here we put $\lambda = |\nabla w|$ and $v_n = (\boldsymbol{n} \bullet \boldsymbol{v})$.

3. An Example

Consider the case when S consists of a sheet that carries only dipoles and currents of dipoles in the empty space. In this case, one has $\rho \equiv \rho_1\delta'(w)$ and $\boldsymbol{J} \equiv \boldsymbol{J}_1\delta'(w)$, which satisfy $\nabla \bullet \boldsymbol{J}_1 + \partial\rho_1/\partial t = 0$. Then, from the compatibility equations (5a–5e) written for $k \geq 2$, one gets first

$$\boldsymbol{E}_k = 0, \qquad \boldsymbol{H}_k = 0, \qquad k = 1, 2, \ldots \quad (6a)$$

Then (5a–5e) written for $k = 1$ give

$$\boldsymbol{E}_0 = \frac{1/\lambda}{1-(v_n/c_0)^2}[\rho_1\boldsymbol{n}/\varepsilon_0 - v_n\mu_0\boldsymbol{J}_1] \quad (6b)$$

$$\boldsymbol{H}_0 = -\frac{1/\lambda}{1-(v_n/c_0)^2}[\boldsymbol{n} \times \boldsymbol{J}_1] \quad (6c)$$

If one inserts these expressions into the boundary conditions (4a–4e), one obtains finally

$$[[\boldsymbol{n} \times \boldsymbol{E}]] = -\frac{1/\lambda}{1-(v_n/c_0)^2}[\Sigma_1 + \Sigma_2] \quad (7a)$$

$$[[\boldsymbol{n} \times \boldsymbol{H}]] = -\frac{1/\lambda}{1-(v_n/c_0)^2}[\Gamma_1 - \Gamma_2] \quad (7b)$$

$$[[\boldsymbol{n} \bullet \boldsymbol{E}]] = -\tfrac{1}{\lambda}\nabla \bullet \boldsymbol{E}_0 \quad (7c)$$

$$[[\boldsymbol{n} \bullet \boldsymbol{H}]] = -\tfrac{1}{\lambda}\nabla \bullet \boldsymbol{H}_0 \quad (7d)$$

$$[[\boldsymbol{n} \bullet \boldsymbol{J}]] = v_n\ [[\rho]] \quad (7e)$$

where we put

$$\Sigma_1 = \nabla \times \boldsymbol{E}_0 + \mu_0\partial\boldsymbol{H}_0/\partial t$$

$$\Sigma_2 = v_n\mu_0[(\nabla \bullet \boldsymbol{H}_0)\boldsymbol{n} - \boldsymbol{n} \times (\nabla \times \boldsymbol{H}_0) + \varepsilon_0\partial(\boldsymbol{n} \times \boldsymbol{E}_0)/\partial t]$$

and

$$\Gamma_1 = \nabla \times \boldsymbol{H}_0 - \varepsilon_0\partial\boldsymbol{E}_0/\partial t,$$

$$\Gamma_2 = v_n\varepsilon_0[(\nabla \bullet \boldsymbol{E}_0)\boldsymbol{n} - \boldsymbol{n} \times (\nabla \times \boldsymbol{E}_0) - \mu_0\partial(\boldsymbol{n} \times \boldsymbol{H}_0)/\partial t]$$

These relations are the boundary conditions satisfied on a moving double sheet.

3.1. A particular case: Monochromatic current on a sheet at rest, Van Bladel's case

Now assume that the sheet is at rest and the current is time harmonic with time factor $\exp\{j\omega t\}$ while the coordinate system (u, v, w) is such that $\lambda = 1$. Then one has $v_n = 0$ and $\partial/\partial t \to j\omega$, which yields

$$\boldsymbol{E}_0 = -(\nabla \bullet \boldsymbol{J}_1)\boldsymbol{n}/(j\omega\varepsilon_0), \ \ \boldsymbol{H}_0 = -\boldsymbol{n} \times \boldsymbol{J}_1 \quad (8)$$

$$[[\boldsymbol{n} \times \boldsymbol{E}]] = \nabla \times [(\nabla \bullet \boldsymbol{J}_1)\boldsymbol{n}]/(j\omega\varepsilon_0) + j\omega\mu_0\boldsymbol{n} \times \boldsymbol{J}_1 \quad (9a)$$

$$[[\boldsymbol{n} \times \boldsymbol{H}]] = \nabla \times (\boldsymbol{n} \times \boldsymbol{J}_1) - (\nabla \bullet \boldsymbol{J}_1)\boldsymbol{n} \quad (9b)$$

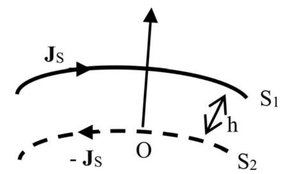

Figure 2. Van Bladel's double sheet. $\boldsymbol{J} = \boldsymbol{J}_S\delta(w - h) - \boldsymbol{J}_S\delta(w)$.

$$[[\boldsymbol{n} \bullet \boldsymbol{E}]] = \nabla \bullet [(\nabla \bullet \boldsymbol{J}_1)\boldsymbol{n}/(j\omega\varepsilon_0)] \tag{9c}$$

$$[[\boldsymbol{n} \bullet \boldsymbol{H}]] = \nabla \bullet (\boldsymbol{n} \times \boldsymbol{J}_1) \tag{9d}$$

It is interesting and important to notice that, without recourse to the above-mentioned distribution postulate, Van Bladel had obtained quite correctly the results given in (8), (9a), and (9d) through an intuitive asymptotic approach. But his approach could not permit him to find the relations given in (9b) and (9c). Now I would like to present the essential points of his approach.

4. Van Bladel's approach

Van Bladel had considered the double current sheet as the limit case of the layer depicted in Figure 2 as $h \to 0$ [8]. The result was obviously the sheet carrying the current

$$\boldsymbol{J} = -\boldsymbol{C}\delta'(w), \qquad \boldsymbol{C} = \lim_{h\to 0}(\boldsymbol{J}_S h) \equiv -\boldsymbol{J}_1, \tag{10}$$

which was equivalent to the problem considered in Section 3.1. His aim was the revelation of the two boundary conditions given in (9a) and (9d). Although he did not state it clearly, his approach depended on a method that required only the use of the dominant singular terms. To this end, he started from the equation $\nabla \times \boldsymbol{H} = \boldsymbol{J} + \partial\boldsymbol{D}/\partial t$written explicitly in the above-mentioned orthogonal coordinate system (u, v, w). He tacitly admitted that the coordinate system is such that the metric coefficients are (h_1, h_2, 1). The assumption $h_3 = 1$ yields $\lambda = |\nabla w| = 1/h_3 = 1$, which means that his case is equivalent to the particular case considered just before. Thus, he writes

$$\frac{\partial H_3}{h_2\partial v} - \frac{\partial h_2}{h_2\partial w}H_2 - \frac{\partial H_2}{\partial w} = j\omega\varepsilon_0 E_1 - C_1\frac{d\delta(w)}{dw} \tag{11a}$$

$$\frac{\partial H_3}{h_1\partial u} - \frac{\partial h_1}{h_1\partial w}H_1 - \frac{\partial H_1}{\partial w} = -j\omega\varepsilon_0 E_2 + C_2\frac{d\delta(w)}{dw} \tag{11b}$$

$$\frac{1}{h_1h_2}\left[\frac{\partial}{\partial u}(h_2H_2) - \frac{\partial}{\partial v}(h_1H_1)\right] = j\omega\varepsilon_0 E_3 \tag{11c}$$

The existence of the function $\delta(w)$ hints that these equations are valid only for a configuration similar to that shown in Figure 1. But from time to time, he also says, "From eqns. (11a–11c) it may be deduced that $\boldsymbol{H}_{\text{tang}}$ and E_3 become infinite *in the layer*, and "Integrating *over the depth of the sheet*," and so on.

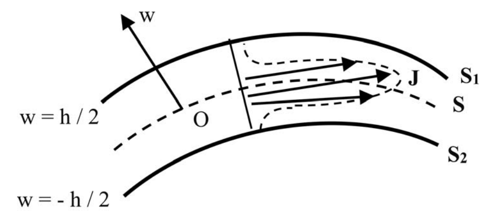

Figure 3. Changes in the Van Bladel's double sheet.

These kinds of expressions, which seem at first glance to be meaningless for the sheet shown in Figure 1, hint that his reasoning oscillates between the double sheet shown in Figure 1($h = 0$) and the layer in Figure 2 ($h > 0$). Hence, his approach needs some elucidation, which I will now attempt.

In order to elucidate his intuition, I consider the following changes in Figure 2:

1) The coordinate surface $w = 0$ is located in the midst of the layer. Thus, $w \in (-h/2,\ h/2)$ (Figure 3).
2) The function $\delta(w)$ is replaced by a continuous function $\delta(w, h)$ which tends to $\delta(w)$ when $h \to 0$. For example, $\delta(w, h)$ may be

$$\delta(w,h) = \frac{1}{h\sqrt{\pi}}e^{-(w/h)^2} \tag{12}$$

Thus, his original equations (11a–11c) become valid in the layer $(-h/2) < w < (h/2)$ with $\delta(w)$ replaced by $\delta(w, h)$.
Then, if we follow his intuitive reasoning, we equate the last terms in the left-hand sides of (11a) and (11b) to the last terms taking place in the right-hand sides, which yields

$$h \to 0 \Rightarrow \boldsymbol{n} \times \boldsymbol{H} = -\boldsymbol{C}\ \delta(w,h) + O(h) \tag{13}$$

The support of this equality is his intuitive guess "the tangential components of the magnetic field tend to infinity when $h \to 0$." Notice that in writing this equality, he ignored the second terms in (11a–11b) , which also tend to infinity as $h \to 0$. If he did not ignore this, then he could not get (13). Although he did not explain the reason for his ignoring this, I think that he tacitly admitted that when $h \to 0$, one has $H_2 = O(\delta(w))$ while $\partial H_2/\partial w = O(\delta'(w))$, and the latter is much more singular than the first. This difference in the order of singularity is obviously seen from the expression of $\delta(w, h)$ in (12).

Now let us return to (13) and use it in (11c) to get

$$h \to 0 \Rightarrow E_3 = \delta(w,h)\ \nabla \bullet \boldsymbol{C}/(j\omega\varepsilon_0) + O(h) \tag{14}$$

Equations (13) and (14) are nothing but (8), which were obtained by straightforward application of the distribution postulate. Notice that from the equation $\nabla \times \boldsymbol{H} = \boldsymbol{J} + \partial\boldsymbol{D}/\partial t$, he obtained only the singular terms of the lowest order associated with the field.

To get the boundary relations (9a) and (9d), he considered the other curl equation, namely, $\nabla \times \boldsymbol{E} = -j\omega\mu_0 \boldsymbol{H}$, which yields explicitly

$$\frac{\partial}{\partial w}(h_1 E_1) = -j\omega\mu_0 h_1 H_2 + \frac{\partial}{\partial u} E_3 \tag{15a}$$

$$\frac{\partial}{\partial w}(h_2 E_2) = j\omega\mu_0 h_2 H_1 + \frac{\partial}{\partial v} E_3 \tag{15b}$$

$$\frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u}(h_2 E_2) - \frac{\partial}{\partial v}(h_1 E_1)\right] = -j\omega\mu_0 H_3 \tag{15c}$$

Now, in (15a) and (15b), he replaces H_1, H_2, and E_3 by the dominant singular expressions obtained above (i.e., (13) and (14)) and integrates the resulting dominant parts in the layer $(-h/2) < w < (h/2)$ to get

$$\boldsymbol{E}^{+}_{\tan g} - \boldsymbol{E}^{-}_{\tan g} = \frac{1}{j\omega\varepsilon_0}\left[\nabla(\nabla \bullet \boldsymbol{C}) + k^2 \boldsymbol{C}\right] \tag{16a}$$

By a straightforward computation, one can show that this is equivalent to (9a).

Finally, to obtain (9d), he differentiates first (15c) (multiplied by $(h_1 h_2)$!) with respect to w and then uses (15a) and (15b) to obtain the derivative of $(h_1 h_2 H_3)$. Integration of this expression in the layer gives

$$\begin{aligned} H_3^{+} - H_3^{-} &= \left[\frac{1}{h_1 h_2}\frac{\partial}{\partial u}(h_2 C_2) - \frac{1}{h_2 h_1}\frac{\partial}{\partial v}(h_1 C_1)\right]_S \\ &= -\nabla \bullet (\boldsymbol{n} \times \boldsymbol{C}) \end{aligned} \tag{16b}$$

This is nothing but (9d).

I would finally like to draw the attention to the fact that in his work, Van Bladel omitted the conditions given by (9b) and (9c). It is perhaps due to his claim that $\boldsymbol{H}_{\text{tang}} = \infty$ and $E_3 = \infty$ for $h \to 0$. In this case, (9b) and (9c) may have seemed meaningless to him because they give rise to indeterminate forms like $(\infty - \infty)$. But as (9b) and (9c) show, these indeterminate forms have well-defined finite values.

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