Determination of the Thresholds in the Split-Step Wavelet Method to Assess Accuracy for Long-Range Propagation

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Abstract – The split-step wavelet method is a method for computing the tropospheric long-range propagation of electromagnetic waves. It follows the same steps as the split-step Fourier method, but the propagation is performed in the wavelet domain instead of the Fourier domain. The efficiency of this method is based on the fast wavelet transform of low complexity and on sparse representation using compression. Nevertheless, the compression introduces an error that accumulates with iteration. In this article, we propose a closed-form formula for the error that allows a priori computation of the compression thresholds for a given scenario and accuracy. Numerical experiments show the relevance of the proposed approach.

1. Introduction

Tropospheric long-range propagation is a topic of major interest for numerous applications in communication, surveillance, and navigation. The parabolic wave equation model [1] is widely used in this context. In the literature, the parabolic wave equation is iteratively solved with the split-step Fourier method (SSF) by going back and forth in the spectral and spatial domains [2, 3]. This method allows large steps to be made in the propagation direction [1].

To accelerate the computation, wavelet-based methods have been proposed in optics [4] and electromagnetics [5–7]. Recently, an efficient split-step wavelet method (SSW) [6–8] was introduced to solve the parabolic wave equation in electromagnetics in both 2D and 3D. This method computes the field by marching on in distances as SSF, but the propagation is performed in the wavelet domain instead of the spectral domain. Each iteration follows two steps. First, the field is decomposed in the wavelet domain and compressed, introducing the signal compression error (threshold \( V_z \)). Second, the coefficients are propagated using a compressed wavelet-to-wavelet propagator, introducing the propagator compression error (threshold \( V_p \)). These errors accumulate throughout the propagation and need to be quantified. Indeed, Zhou et al. [6] have shown that SSW is faster than SSF if good compression is performed.

The main contribution of this article is that we derive a theoretical formula of the accumulated compression error after \( N_z \) iterations. This allows us to set both thresholds a priori for a given accuracy and scenario. We also show that the heuristic formula proposed in earlier works [6, 9] is too optimistic.

For conciseness, the method and the proof are developed in 2D. Nevertheless, the results remain valid in 3D using similar calculations. After a general presentation of SSW (Section 2), the error formula is derived (Section 3) and tested via numerical experiments (Section 4) in 2D.

2. Overview of the Split-Step Wavelet Method

2.1 Configuration and Discretization

In this article, an \( \exp(j \omega t) \) time dependence is assumed, where \( \omega \) is the angular frequency. The domain is 2D, of size \([0, x_{\text{max}}]\) in \( x \) and \([0, z_{\text{max}}]\) in \( z \). The field is known at \( x = 0 \) and the source is placed at \( x \leq 0 \). On the \( z \)-axis a sampling is made with \( z_p = p_z \Delta z \) and \( p_z \in \{0, \ldots, N_z - 1\} \). On the \( x \)-axis a sampling is made with \( x_p = p_x \Delta x \) and \( p_x \in \{0, \ldots, N_x - 1\} \).

2.2 Brief Reminder of the Discrete Wavelet Transform

The wavelet family is computed by dilating and translating a mother wavelet of zero mean on \( L \) levels [10]. Dilations allow covering of the lower parts of the spectrum. To obtain an orthonormal basis, a scaling function of nonzero mean is added. This function covers the lowest part of the spectrum. Using this basis, a multilevel decomposition is obtained. We recall one important property of the wavelets used in the following demonstrations. The number of vanishing moments \( n_v \) of a wavelet \( \psi \) is defined as

\[
\forall k \in [0, n_v], \quad \int z^k \psi(z)dz = 0
\]  

(1)

This property describes how well a wavelet decomposition can approach a smooth function with few coefficients.
2.3 An Overview of SSW

SSW computes the field iteratively by going back and forth from the wavelet domain to the spatial domain. As with SSF, the refraction and relief are taken into account in the spatial domain, and thus we describe only the free-space propagation part. The initial field \( u_0 \) is supposed known.

Denoting by \( u_p \) the field and by \( U_p \), its wavelet decomposition at distance \( p, \Delta \), a step of SSW is computed as follows. First the wavelet coefficients \( U_p \) are computed by applying a fast wavelet transform on \( u_p \) (denoted \( \mathbf{W} \)). Then the coefficients are compressed (operator \( C_{V_s} \)) with a hard threshold \( V_s \) (i.e., all coefficients below \( V_s \) are set to 0). A sparse vector of size \( N_s \) is obtained. This first compression repeated on \( N_s \) horizontal steps induces an error term denoted by \( \delta_{N_s}^e \).

Then the coefficients are propagated using a precomputed matrix \( \mathbf{P} \):

\[
U_{p+1} = \mathbf{PC}_{V_s} U_p
\]

This sparse matrix contains all the wavelet-to-wavelet propagations and is of size \( (N_s, N_s) \) [6]. A compression with hard threshold \( V_s \) is performed. Iterated \( N_s \) times, this second error term in the method is denoted by \( \delta_{N_s}^m \). The free-space propagated field is obtained by coming back in the space domain using an inverse fast wavelet transform. The total compression error is denoted by \( \delta_{N_s} = \delta_{N_s}^e + \delta_{N_s}^m \).

In [6] the experimental upper bound was supposed to follow \( N_s^{-2} V_s \) and \( N_s V_s \) for the signal and propagator compressions, respectively. We derive here new and more accurate expressions.

3. Derivation of the Compression-Error Formula

In the following section, we introduce the normalized thresholds \( v_s \) and \( v_p \) such that \( V_s = v_s \| U_0 \|_\infty \) and \( V_p = v_p \| \mathbf{P} \|_{\infty} \) [10, 11]. We also note that the operator norm of \( \mathbf{P} \) corresponds to

\[
\| \mathbf{P} \|_{\infty} = \sup_{U \neq 0} \| \mathbf{P} U \|_2 / \| U \|_2
\]

From power conservation, the operator norm of the free-space propagator \( \mathbf{P} \) is equal to 1 (\( \| \mathbf{P} \|_{\infty} = 1 \)). If there are no evanescent waves and the propagation does not reach any boundaries, then we have \( \| \mathbf{P} u \|_2 = \| u \|_2 \). In other cases (apodization, environment losses, evanescent waves, and so on), then we have \( \| \mathbf{P} u \|_2 \leq \| u \|_2 \).

3.1 Signal Compression Error

The objective of this section is to study how the signal compression error accumulates with \( N_s \). We first assume that \( V_s = 0 \) and \( V_p = 0 \). The propagator has no compression. The error due to the threshold \( V_s \) on the signal (operator \( C_{V_s} \)) after \( N_s \) iterations is defined by

\[
\delta_{N_s}^e = \| \hat{U}_{N_s} - U_{N_s} \|_2 / \| U_0 \|_2
\]

with \( \hat{U}_{N_s} = (\mathbf{PC}_{V_s})^N U_0 \) and \( U_{N_s} = \mathbf{P}^N U_0 \) the compressed and uncompressed propagated coefficients, respectively.

For one iteration, the error is given by

\[
\delta_1^e = \| \mathbf{PC}_{V_s} U_0 - U_0 \|_2 / \| U_0 \|_2
\]

We introduce \( \varepsilon_0 \), the compression term due to \( C_{V_s} \), defined by

\[
C_{V_s} U_0 = U_0 + \varepsilon_0
\]

Using (3) and introducing (6) into (5), we obtain \( \delta_1^e \leq \| \varepsilon_0 \|_2 / \| U_0 \|_2 \). For the smooth signals we are manipulating, the wavelet coefficients decrease exponentially to 0 [10, 11]. Therefore, we rewrite the norm of the error as

\[
\| \varepsilon \|_2 = v_s^2 \| U_0 \|_\infty^2 \sum_{m=0}^{N_s-M-1} |\varepsilon_m^0|^2
\]

with the coefficients \( |\varepsilon_m^0| \leq 1 \) corresponding to the normalized amplitudes of the wavelet coefficients of the error indexed in decreasing order, and \( M \ll N_s \) the number of significant coefficients. Following [10–12], error components are bounded by

\[
|\varepsilon_m| \leq C_v (m+1)^{-n_v}
\]

with \( m \in [0, N_s - M - 1] \), \( n_v \) the number of vanishing moments, and \( C_v \) a constant depending only on the smoothness of the field and of the wavelets. Putting (8) in (7), an upper bound for \( \| \varepsilon_0 \|_2 \) is obtained:

\[
\| \varepsilon_0 \|_2 \leq v_s \| U_0 \|_\infty C_v \sqrt{\sum_{m=0}^{N_s-M-1} (m+1)^{-2n_v}}
\]

For \( n_v \geq 2 \), the sum converges close to 1 (e.g., for \( n_v = 2 \) it is about 1.082). Also, \( C_v \) is less than or close to 1, as illustrated with numerous numerical tests in Section 4 and in [13].

Thus, the bound on the error due to signal compression after one iteration is given by

\[
\| \varepsilon_0 \|_2 \leq v_s \| U_0 \|_2 \text{ and } \delta_1^e \leq v_s
\]

where \( \leq \) means less than or close to, as widely used in the wavelet community [10, 11]. In practice, this result shows very good accuracy, with numerous numerical tests performed in [6, 14].

For two iterations, we compare the propagations with and without compression:

\[
\delta_2^e = \| (\mathbf{PC}_{V_s}) U_0 - \mathbf{PP} U_0 \|_2 / \| U_0 \|_2
\]

We define the second compression error \( \varepsilon_1 \) as

\[
C_{V_s}(\mathbf{P} U_0 + \varepsilon_0) = \mathbf{P} U_0 + \mathbf{P} \varepsilon_0 + \varepsilon_1
\]

The expression of the error is calculated as
\[ \delta_d^2 = \| PP U_0 + PP e_0 + P e_1 - PP U_0 \|_2^2 / \| U_0 \|_2^2 \leq (\| e_0 \|_2^2 + \| e_1 \|_2^2) / \| U_0 \|_2^2 \]  

(12)

Supposing (10) is true for the second iteration, we obtain

\[ \| e_1 \|_2^2 / \| U_0 \|_2^2 \leq v_s \left( \| U_0 \|_2^2 + \| e_0 \|_2^2 \right) / \| U_0 \|_2^2 \]  

(13)

Since, with an appropriate threshold, \( \| e_0 \|_2 \) is negligible to \( \| U_0 \|_2 \), we have \( \delta_d^2 \leq 2v_s \). By induction, the signal compression error after \( N_x \) horizontal iterations fulfills

\[ \delta_{N_x}^2 \leq N_x v_s \]  

(14)

The appropriate threshold \( v_s = v_s / \| U_0 \|_\infty \) can now be computed with (14). The same study is now performed on the error due to the compression of the propagator \( P \) in (2).

### 3.2 Propagator Compression Error

We now assume that \( v_s = 0 \) and \( v_p = 0 \). The error \( \delta_{N_x}^p \), due to the compression of the propagator after \( N_x \) iterations is studied. It is defined by

\[ \delta_{N_x}^p = \| \tilde{U}_{N_x} - U_{N_x} \|_2^2 / \| U_0 \|_2^2 \]  

(15)

where \( \tilde{U}_{N_x} \) corresponds to the coefficients propagated \( N_x \) times with the compressed operator denoted as \( P + \Delta P \).

From [11, pp. 29–32], we have the norm operator of \( \Delta P \) bounded by

\[ \| \Delta P \|_{op} = \sup_{U \neq 0} \| \Delta P U \|_2 / \| U \|_2 \leq v_p \]  

(16)

For one iteration, the expression of the error is given by

\[ \delta^p = \| \tilde{U}_1 - U_1 \|_2^2 / \| U_0 \|_2^2 = \| \Delta P U_0 \|_2^2 / \| U_0 \|_2^2 \]  

(17)

Following (16), we have \( \delta^p \leq v_p \) in agreement with Kremp and Freude [4].

Using the same notations and methodology as for one iteration, and since \( \| P \|_{op} = 1 \), using (16) we obtain for two iterations

\[ \delta_{N_x}^p = \| \tilde{U}_2 - U_2 \|_2^2 / \| U_0 \|_2^2 \leq \| \Delta PP U_0 \|_2^2 + \| \Delta P U_0 \|_2^2 + \| \Delta P^2 U_0 \|_2^2 / \| U_0 \|_2^2 \leq 2v_p + v_p^2 \]  

(18)

Neglecting the term \( v_p^2 \) (\( v_p \ll 1 \)), we see that \( \delta_{N_x}^p \) is less than or close to \( 2v_p \). By induction, we finally obtain

\[ \delta_{N_x}^p \leq v_p N_x \]  

(19)

From (19), we are able to choose the adequate threshold \( v_x \) for a given error and scenario.

Assuming that both errors are independent, we finally derive a closed-form expression for the accumulated compression error \( \delta_{N_x} \leq (v_s + v_p) N_x \). In practice, for a given maximum expected error \( \delta_{N_x}^{max} \) and number of iterations \( N_x \), the normalized thresholds are computed as

\[ v_s = \delta_{N_x}^{max} / (2N_x) \quad \text{and} \quad v_p = \delta_{N_x}^{max} / (2N_x) \]  

(20)

Thus we derive the unnormalized thresholds

\[ v_s = \frac{\delta_{N_x}^{max}}{2N_x} \| U_0 \|_\infty \quad \text{and} \quad v_p = \frac{\delta_{N_x}^{max}}{2N_x} \| P \|_{\infty} \]  

(21)

### 4. Numerical Tests

In this section, numerical experiments are performed to show that the thresholds \( v_s \) and \( v_p \) can be managed to assess a given final accuracy for a certain number of iterations \( N_x \), using (20). First, a short-range simulation in free space is performed to assess the accuracy of the formulas. Second, we perform a long-range simulation with relief and refraction.

#### 4.1 Free-Space Scenario

We perform the tests in 2D. The source is a uniform aperture at \( f_0 = 300 \text{ MHz} \) of size 10 m placed at \( z_s = 1024 \text{ m} \) in a domain of vertical size \( z_{max} = 2048 \text{ m} \). The domain is of horizontal size \( x_{max} = 2000 \text{ m} \). The step sizes are \( \Delta x = 20 \text{ m} \) and \( \Delta z = 0.5 \text{ m} \). Thus, we have \( N_x = 100 \). For the wavelet parameters, the symlet with \( n_s = 6 \) and a maximum level of \( L = 3 \) is chosen.

The root-mean-square error between compressed and uncompressed propagations is computed for different values of \( N_x \) and compared to the closed-form formulas. Thresholds are set to \( v_s = v_p = 1.6 \times 10^{-4} \) using (20), so as to obtain an error of \( -30 \text{ dB} \) at the final range.

First, as shown in Figure 1, we compute and plot the constant \( C_s \approx \| \epsilon \|_2 / V_s \) at each step \( N_x \). This shows that the constant is less than or close to 1, which corroborates the approximation leading to (14) proposed in Section 3.1.

The root-mean-square error is computed and given in Figure 2. As expected, the closed-form formula for the compression error is never reached. The
computed thresholds allow the error to be bounded below the desired maximum. We also compute a linear regression to find the optimal $a$ such that $d_N > v_a N^2$ and $d_p > v_p N^2$. For the signal compression, we obtain $a = 0.96$, slightly lower than the value proposed here—i.e., 1—but greater than the heuristic value (0.5) proposed in [6]. This shows that the heuristic formula proposed in [6] was too optimistic. For the propagator, we obtain $a = 0.97$, close to the value proposed here and in [6]. This illustrates the relevance of the proposed formulas (20). Therefore, they can be used to tune the thresholds needed in SSW to obtain a given accuracy. In the next section, a numerical test in realistic conditions is performed.

### 4.2 Realistic Scenario

The propagation of a complex source point in a domain with a trilinear atmosphere and two triangular reliefs is computed. The parameters of the complex source point are frequency $f = 300$ MHz, coordinates $x_{0,0} = 50$ m and $z_s = 50$ m, and waist size $W_0 = 5$ m. We consider an atmosphere described by a trilinear duct [15] of base height $z_b = 241$ m, thickness $z_t = 391$ m, and gradient $c_2 = -0.5$ M-units/m in the duct and $c_0 = 0.118$ M-units/m elsewhere. On the ground, we choose $M_0 = 330$ M-units. The relief is chosen as two small triangles of heights 100 m and 200 m. The impedance ground has parameters $\epsilon_r = 20.0$ and $\sigma = 0.02$ S/m.

The domain is of horizontal size $x_{\text{max}} = 100$ km and vertical size $z_{\text{max}} = 2048$ m. An apodization window is added on top of the domain. The grid size is 200 m horizontal and 0.5 m vertical. We aim at obtaining an error of $-30$ dB at the final iteration. From (20) we apply the thresholds $v_s = v_p = 3.16 \times 10^{-5}$.

In Figure 3, the field (in decibel-volts per meter) is plotted in (a) and the root-mean-square error evolution is plotted in (b). We can see that the bound is not reached and that the final error is significantly smaller than the desired error. This is mostly due to the apodization layer, in which energy is leaving the computational domain, reducing the total error. Therefore, our formula is conservative in a realistic domain.

### 4.2 Realistic Scenario

In this article, we have derived a closed-form expression for the accumulated compression error in the split-step wavelet method (SSW). This formula allows the thresholds $V_s$ and $V_p$ to be tuned for a given accuracy.

First, we gave an overview of SSW to show where thresholds are applied. The compressions $V_s$ on the signal and $V_p$ on the propagator introduce errors. We derived how each error accumulates during iteration to obtain a conservative expression for the compression error. This latter allows us to set $V_s$ and $V_p$ a priori for a given accuracy and scenario. Finally, numerical tests in 2D were performed.

To conclude, the expression obtained in this article for the accumulated compression error is now successfully used in SSW [8] to tune $V_s$ and $V_p$ for a given scenario.

### 5. Conclusion

In this article, we have derived a closed-form expression for the accumulated compression error in the split-step wavelet method (SSW). This formula allows the thresholds $V_s$ and $V_p$ to be tuned for a given accuracy.

First, we gave an overview of SSW to show where thresholds are applied. The compressions $V_s$ on the signal and $V_p$ on the propagator introduce errors. We derived how each error accumulates during iteration to obtain a conservative expression for the compression error. This latter allows us to set $V_s$ and $V_p$ a priori for a given accuracy and scenario. Finally, numerical tests in 2D were performed.

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### 6. References

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