Josephson-Effect-Based Electronics

Peter Russer and Johannes A. Russer

Abstract – Josephson-junction circuits allow generation, mixing, detection, and parametric amplification in classical and quantum signal processing. General energy relations for Josephson junctions govern frequency conversion and AC–DC conversion. In this article, we review physical principles, properties, and applications of superconducting electronics based on Josephson parametric amplifiers, and give an overview of the development over the last 50 years.

1. Introduction

Josephson-effect-based devices allow generation, detection, mixing, and parametric amplification of high-frequency signals up into the terahertz region and exhibit high sensitivity, low energy consumption, and small size [1–4]. According to the theory of Bardeen, Cooper, and Schrieffer, superconductivity is a microscopic effect caused by a Bose–Einstein condensation of electrons into Cooper pairs [5]. The superconducting ground state can be described by a coherent macroscopic matter wave function. The condensation of the electrons in a quasi-coherent state leads to special high-frequency properties and low electronic noise. Superconducting tunneling currents were observed experimentally by Isolde Dietrich in 1952 [6]. In 1962, B. D. Josephson presented the theory of superconductive tunneling through superconductor–insulator–superconductor junctions based on the microscopic theory of Bardeen, Cooper, and Schrieffer [8, 9]. The experimental proof of the Josephson effect, confirming important predictions made by Josephson, was given by P. W. Anderson and J. M. Rowell [7].

2. General Energy Relations

A Josephson junction, shown schematically in Figure 1, is an arrangement of two superconductors S1 and S2 weakly coupled across an insulating tunnel barrier TB with a thickness of a few nanometers. A voltage $v(t)$ applied to the Josephson junction determines the time variation of the quantum phase difference φ of the macroscopic matter waves describing the superconducting state. Current $i(t)$ and voltage $v(t)$ are related via

$$i(t) = I_J \sin \varphi(t) \tag{1a}$$

$$v(t) = \frac{\hbar}{2e_0} \frac{\mathrm{d}\varphi(t)}{dt} \tag{1b}$$

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where $\hbar = h/2\pi$, h is Planck's constant, e_0 is the magnitude of the electron charge, and I_J is the maximum Josephson current [2].

Applying a DC voltage V_0 yields an AC Josephson current with frequency $f_0 = 2e_0V_0/h = 483.6V_0$ (MHz/μV) and amplitude I_J. The energy $w_J(\varphi)$ stored in the Josephson junction, shown also in Figure 2, is

$$\begin{aligned} w_J(\varphi) &= \frac{\hbar I_J}{2e_0}[1 - \cos\varphi] \\ &= \frac{1}{2\pi}\Phi_0 I_J[1 - \cos(2\pi\Phi(t)/\Phi_0)] \end{aligned} \tag{2}$$

where Φ is a magnetic flux, defined via $d\Phi/dt = v$, and the flux quantum $\Phi_0 = h/2e_0 \approx 2.06783461 \times 10^{-15}$ Vs. The maximum energy that can be stored in a Josephson junction is given by $\hbar I_J/2e_0 = \Phi_0 I_J/2\pi$, and the maximum power that can be exchanged with a single Josephson junction at a frequency f is less than $f\Phi_0 I_J/2\pi$ [10]. This yields for $I_J = 1$ μA and $f = 10$ GHz a saturation power below 3 pW.

The Josephson junction acts as a nonlinear lossless inductor with fundamental inductance $L_J = \hbar/2e_0 I_J$ and can be applied for mixing and parametric amplification in the microwave region. The general energy relations for the Josephson junction [11, 12] are similar to the Manley–Rowe equations [13], however, they exhibit an additional term for the DC power.

In the equivalent circuit [2] depicted in Figure 3, power exchange with the Josephson junction is only possible at DC and at the combination frequencies $mf_1 + nf_2$, where m and n are integers. Applying a voltage with a DC component V_0 and AC components with frequencies f_1 and f_2 yields a Josephson current with frequency components $f_0 + mf_1 + nf_2$, where m, n are integers. For the case $f_0 = kf_1 + lf_2$, the following general energy relations were derived in [12]:

$$\sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{mP_{mn}}{mf_1 + nf_2} = -\frac{kP_0}{kf_1 + lf_2} \tag{3a}$$

$$\sum_{n=-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{mP_{mn}}{mf_1 + nf_2} = -\frac{lP_0}{kf_1 + lf_2} \tag{3b}$$

where P_{mn} is the active power flowing into the Josephson junction at the combination frequency $mf_1 + nf_2$ and P_0 is the DC power flowing into the Josephson junction. If the Josephson junction is connected only to two resonant circuits with frequencies f_1 and f_2, and the junction is DC voltage-biased to generate a Josephson oscillation at $f_0 = f_1 + f_2$, we obtain $P_1/f_1 = P_2/f_2 = -P_0/(f_1 + f_2)$.

Figure 1. Josephson junction.

3. Josephson Parametric Amplifiers

The first experimental realization of an AC-pumped Josephson parametric amplifier (ACPJPA) was reported in 1967 by H. Zimmer [14]. This ACPJPA consisted of a thin-film Josephson junction evaporated on a rutile resonator and was operated at 9646 MHz in the double-degenerate mode.

In 1969, P. Russer proposed a DC-pumped Josephson parametric amplifier (DCPJPA) [11]. Figure 4 shows the equivalent circuit of the DCPJPA exhibiting a signal circuit consisting of the inductor L_{10}, the capacitor C_{10}, the conductor G_{10}, and the impressed signal source I_{10}, as well as an idler circuit consisting of L_{01}, C_{01}, and G_{01}. Terminating the Josephson junction at the idler frequency ω_2 with the admittance Y_{01}, it exhibits the impedance

$$Z_{J10} = -\frac{\omega_1 \omega_2 \hbar^2}{e_0^2 I_J^2} Y_{01}^* \tag{4}$$

at the signal frequency ω_1. Since the negative real part of Z_{J10} is related to the positive real part of Y_{01}, the gain at ω_1 is related to losses in the idler circuit. The magnitudes of the transfer admittances of the Josephson junction governing the conversion between signal and idler frequencies are given by $|Y_{12}| = e_0 I_J/\hbar\omega_2$ and $|Y_{21}| = e_0 I_J/\hbar\omega_1$ and determine the admittance level of the circuitry [11].

H. Kanter has realized DCPJPAs for signal frequencies of 30 MHz [15] and 9 GHz [16–18], as up-converters from 115 MHz to 9 GHz [17, 19], and for 89 GHz [20]. A 15 GHz mixer with niobium point contact is described in [21]. M. Yu proposed a DCPJPA where the Josephson oscillation synchronizes with the input signal [22]. With a DCPJPA, Calander, Claeson, and Rudner achieved 8 dB gain and 2 pW input saturation level at 10 GHz [23]. With an ACPJPA operating at 9 GHz, a power gain of 16 dB in a 4 MHz, 3 dB bandwidth was achieved by J. Mygind et al. [24, 25]. A JPA for 36 GHz with a point-contact Josephson junction with 11 dB gain was realized by Y. Taur and P. Richards [26, 27]. Further work on JPAs is presented in [10, 28–37].

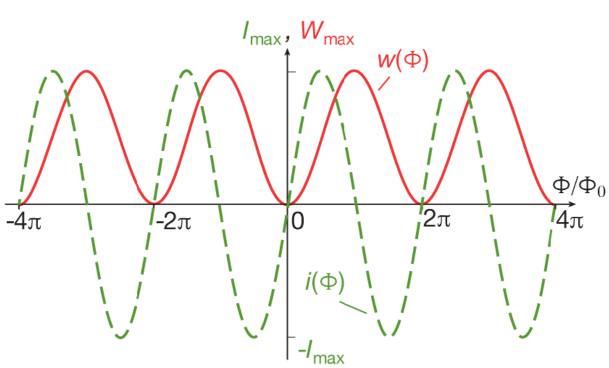

Figure 2. Current $i(\Phi)$ flowing through the Josephson junction and energy $w(\Phi)$ stored in the junction.

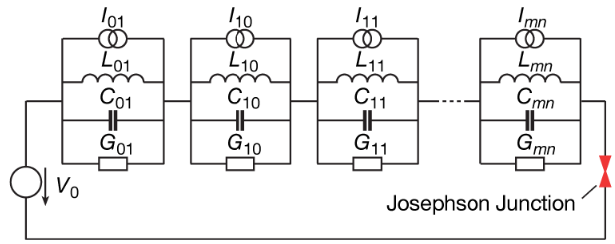

Figure 3. Frequency-conversion circuitry [2].

SQUID JPAs use Josephson junctions in a superconducting quantum interference device arrangement, with one or two Josephson junctions inserted in a superconducting ring. For a single junction, the Josephson current is given by (1a) for $\varphi = 2\pi\Phi/\Phi_0$, where Φ denotes the magnetic flux through the ring [32, 38, 39]. Mutus et al. reported a SQUID JPA tunable between 5 GHz and 7 GHz with quantum-limited noise performance, and an input saturation power greater than 120 dBm [40]. By proper impedance matching with a flux-pumped amplifier made of an array of RF-SQUIDs with 40 Josephson junctions, a high saturation power of –90 dBm and a bandwidth greater than 1.6 GHz have been achieved [41]. Planat et al. realized a JPA consisting of an array of 80 SQUIDs exhibiting a bandwidth of 45 MHz, tunable between 5.9 GHz and 6.8 GHz, with an input saturation power of –117 dBm [42].

4. Traveling-Wave Parametric Amplifiers

A Josephson traveling-wave parametric amplifier (JTWPA) achieves unilateral and higher gain, together with improved stability and larger bandwidth, than a discrete JPA with a single Josephson junction [2, 43–51]. The JTWPA can be realized either by connecting Josephson junctions in parallel to a transmission line or by inserting Josephson elements in series between transmission-line segments.

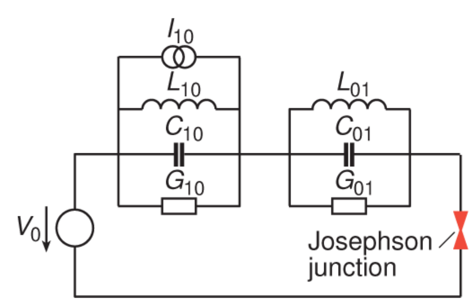

Figure 4. Josephson parametric amplifier [11].

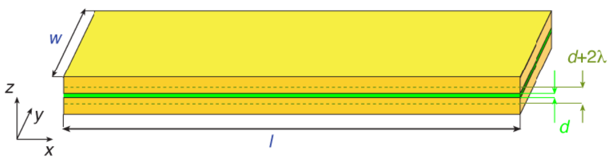

Figure 5. Josephson transmission line.

Figure 5 shows a JTWPA structure with the Josephson junction in parallel to the transmission line, as described in [43, 44]. The JTWPA is based on a Josephson transmission line formed by two superconductors of length l connected via a distributed Josephson junction exhibiting an insulating tunneling layer of thickness d. A transverse magnetic field B_y flows through a layer of thickness $d_m = d + 2\lambda_L$, where λ_L is the London penetration depth of the magnetic field, into the superconductor [52, p. 61]. The magnetic field yields a spatial variation of the quantum phase difference φ [53].

For a uniform transverse magnetic field B_{y0}, the magnetic flux increases in the x-direction proportional to $x d_m B_{y0}$. Applying a DC voltage V_0 and a static transverse magnetic field B_{y0}, we excite a pump wave with angular frequency $\omega_0 = 2e_0 V_0/\hbar$ and wave number $k_0 = 2e_0 B_{y0} d_m/\hbar$. For the applied idler frequencies $\omega_{i\pm}$, both frequency and phase conditions $\omega_{i\pm} = \omega_0 \pm \omega_s$ and $k_{i\pm} = k_0 \pm k_s$ must be fulfilled. Small signal analysis [2, 43, 44] yields a power gain at ω_s given as

$$G_s(x) = \frac{1}{2}[\cosh(2\kappa l) + 1],$$
$$\kappa = \frac{1}{4\lambda_j^2 \sqrt{2k_{i-}k_{i+}}} = \frac{1}{4\lambda_j^2 \sqrt{2(k_0^2 - {k_s}^2)}} \tag{5}$$

where J_J is the maximum Josephson current density and $k_{i\pm}$ and k_s are the wave numbers of the idler and signal waves, respectively.

The JTWPA with Josephson junctions in parallel to the transmission line has the advantages that it can be realized as a continuous structure, it can be DC-pumped, and the phase velocity of the pump wave can be tuned by a DC magnetic field. Difficulties for realization arise from the low impedance of Josephson junctions.

In 1985, M. Sweeny and R. Mahler proposed a JTWPA consisting of a superconducting transmission line interrupted in series by a large number of junctions [45]. Similar structures with series-connected Josephson junctions were investigated in [46–48, 50, 51, 54]. Figure 6 shows the circuit of a Josephson traveling-wave parametric amplifier consisting of a long chain of capacitively shunted Josephson junctions along a transmission line.

Naaman, Ferguson, and Epstein presented the design of a three-wave JTWPA with a bandwidth of 1.6 GHz and a high saturation power of –90 dBm at a gain of 22.8 dB [41]. With a JTWPA based on a periodic so-called photonic crystal structure formed by 2160 SQUIDs, Planat et al. achieved 3 GHz bandwidth around 6 GHz, –100 dBm saturation power with a noise figure near the quantum limit [55].

5. Circuit Quantum Electrodynamics of Josephson Parametric Amplifiers

Given small enough signal amplitudes, a treatment on the basis of circuit quantum electrodynamics (CQED) is required in order to obtain an understanding and a correct description of the phenomena, taking into account the quantum statistical properties of the circuits. CQED allows for an investigation of lumped-element or distributed linear lossless circuits on the basis of Hamiltonian description of Foster equivalent circuits [56–60].

The quantum theory of circuits has already been addressed by H. A. Haus and Y. Yamamoto [56] and by B. Yurke [57]. QCED models can be constructed for electromagnetic structures using analytic or numerical methods in conjunction with system identification methods, yielding lumped element models in canonical form. Figure 7 shows a Josephson junction embedded in a circuit consisting of two lossless resonant circuits L_1, C_1 and L_2, C_2, and a DC source V_0. Introducing annihilation operators $\boldsymbol{a}_i$ and creation operators $\boldsymbol{a}_i^\dagger$ by

$$\boldsymbol{a}_i = \sqrt{\frac{1}{2\hbar\omega_i L_i}}\boldsymbol{\Phi}_i + j\sqrt{\frac{\omega_i L_i}{2\hbar}}\boldsymbol{Q}_i \tag{6a}$$

$$\boldsymbol{a}_i^\dagger = \sqrt{\frac{1}{2\hbar\omega_i L_i}}\boldsymbol{\Phi}_i - j\sqrt{\frac{\omega_i L_i}{2\hbar}}\boldsymbol{Q}_i \tag{6b}$$

where the operators $\boldsymbol{Q}_i$ and $\boldsymbol{\Phi}_i$ represent charge and flux in the capacitors C_i and the inductors L_i, respectively. The Hamiltonian $\boldsymbol{H}$ is given by

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_1 \tag{7}$$

$$\boldsymbol{H}_0 = \frac{1}{2}\hbar\omega_1(\boldsymbol{a}_1^\dagger\boldsymbol{a}_1 + \boldsymbol{a}_1\boldsymbol{a}_1^\dagger) + \frac{1}{2}\hbar\omega_2(\boldsymbol{a}_2^\dagger\boldsymbol{a}_2 + \boldsymbol{a}_2\boldsymbol{a}_2^\dagger) \tag{8}$$

$$\boldsymbol{H}_1 = W_J\left[1 - \cos\left[\omega_0 t + \kappa_1\left(\boldsymbol{a}_1 + \boldsymbol{a}_1^\dagger\right) + \kappa_2\left(\boldsymbol{a}_2 + \boldsymbol{a}_2^\dagger\right)\right]\right]$$

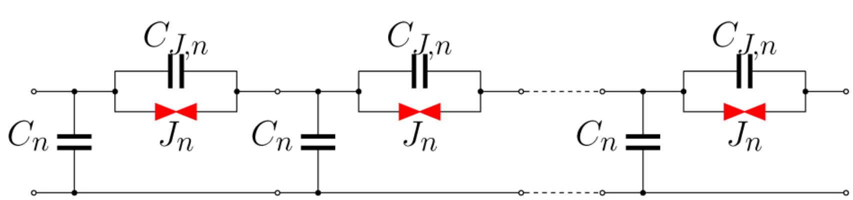

Figure 6. JTWPA with series-connected junctions.

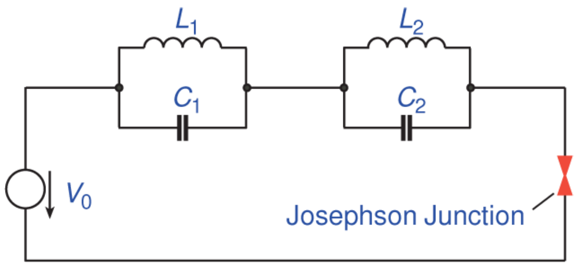

Figure 7. Josephson-junction circuit.

with $\omega_0 = \frac{2e_0 V_0}{\hbar}$, $\Phi_0 = \frac{\pi\hbar}{e_0}$, $W_J = \frac{\Phi_0 I_J}{2\pi}$, $\kappa_i = \sqrt{\frac{2\alpha Z_i}{\pi Z_{F0}}}$, $Z_{F0} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \approx 377\Omega$, $Z_i = \sqrt{L_i/C_i}$.

The element α is the fine structure constant $\alpha = e_0^2 Z_{F0}/4\pi\hbar \approx 1/137$. The time dependence of the expectation value of the photon energy has been calculated in [2, 59] as

$$\begin{aligned}\langle W(t)\rangle &= \hbar\omega|w|^2\cosh^2\gamma_{12}t \\ &+ \frac{\hbar\omega_1}{2}\coth\frac{\hbar\omega_1}{k_B T_1}\cosh^2\gamma_{12}t \\ &+ \frac{\hbar\omega_1}{2}\coth\frac{\hbar\omega_2}{k_B T_2}\sinh^2\gamma_{12}t \end{aligned} \tag{9}$$

with

$$\gamma_{ij} = \frac{e_0 I_J}{\pi h}\sqrt{Z_i Z_j} = \frac{\alpha I_J}{\pi e_0}\frac{\sqrt{Z_i Z_j}}{Z_{F0}} = \frac{I_J}{2\pi\Phi_0}\sqrt{Z_i Z_j} \tag{10}$$

The first term on the right-hand side of (9) represents the amplified signal, the second term is the amplified noise of the signal circuit L_1, C_1, and the third term is the amplified noise down-converted from the idler circuit L_2, C_2 to the signal circuit. A quantum-mechanical treatment of the JTWPA was given in [61, 62]. Dissipation and fluctuation in DCPJPAs were investigated in [63–66] on the basis of the quantum Langevin equations. Roy and Devoret [67] and Sivak et al. [68, 69] discussed the influence of the Kerr terms $\sim \boldsymbol{a}^{\dagger 2}\boldsymbol{a}^2$ on the dynamic range of JPAs.

6. Squeezed States and Entangled States

Squeezed states have less uncertainty in one quadrature than coherent states [70–72]. Squeezed states of the radiation field are eigenstates of the operator

$$\boldsymbol{b} = \mu\boldsymbol{a} + \nu\boldsymbol{a}^\dagger, \quad |\mu|^2 - |\nu|^2 = 1 \tag{11}$$

where $\boldsymbol{a}^\dagger$ and $\boldsymbol{a}$ are photon creation and annihilation operators and and ν are complex numbers. Splitting $\boldsymbol{a}$ into $\boldsymbol{a} = \boldsymbol{a}_c + j\boldsymbol{a}_q$, representing cophasal and quadrature components, with $\boldsymbol{a}_c = \boldsymbol{a}_c^\dagger$ and $\boldsymbol{a}_q = \boldsymbol{a}_q^\dagger$, yields

$$\left\langle \Delta\boldsymbol{a}_c^2 \right\rangle = \frac{1}{4}|\mu - \nu|^2, \quad \left\langle \Delta\boldsymbol{a}_q^2 \right\rangle = \frac{1}{4}|\mu + \nu|^2 \tag{12}$$

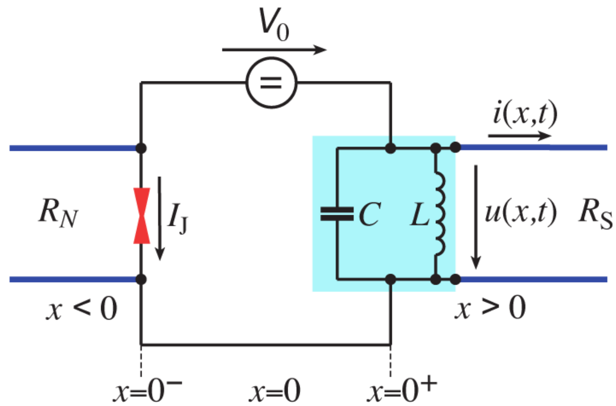

Figure 8. Schematic circuit diagram of the DCPJPA [77].

Squeezed states can be generated by degenerate parametric amplification [73]. Squeezed state generation using a Josephson parametric amplifier has been discussed in [74–77]. The achievable degree of squeezing in a degenerated DCPJPA with identical signal and idler frequencies $f_1 = f_2 = f_0/2$ has been calculated in [76, 77]. Figure 8 shows the schematic circuit diagram of the DCPJPA. The Josephson junction is DC-biased by a voltage V_0, connected to the resonant circuit LC, and resistively shunted by R_N. The DC bias V_0 is chosen such that the frequency f_0 is twice the resonance frequency of the resonant circuit LC [76], and (8) can be approximated by

$$\boldsymbol{H}_1^{\mathrm{DCPJPA}} = \kappa_1\left[\boldsymbol{a}^{\dagger 2}e^{-j\omega_0 t} + \boldsymbol{a}^2 e^{j\omega_0 t}\right] \tag{13}$$

According to Yuen, this Hamiltonian can produce squeezed states [70–72]. Squeezed states allow transfer of entanglement to a pair of quantum bits [78]. Squeezing of photons with JPAs was investigated in [62, 79, 80].

Josephson junctions allow manipulation of quantum information. The qubit (quantum bit) is the basic unit of quantum information and is represented by a two-state quantum system [81]. While a classical bit can assume only either the states 0 or 1, a qubit can be in a weighted superposition of states $|0\rangle$ and $|1\rangle$. A two-qubit state which is not decomposable into two one-bit states is called entangled. An example of an entangled state is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

A photonic NOON state is a many-photon entangled state representing a superposition of N particles in a first mode a, with zero particles in a second mode b, and vice versa [82, 83]; it has the form

$$|\psi\rangle_{\mathrm{NOON}} = \frac{1}{\sqrt{2}}\left(|N,0\rangle + e^{j\phi}|0,N\rangle\right) \tag{14}$$

NOON states allow us to make precision phase measurements in optical interferometry. NOON-state generation with Josephson-junction circuits is reported in [84, 85].

7. Applications

JPAs are of fundamental importance as components of superconducting quantum information processing systems and as interfaces to conventional electronics.

Josephson circuits allow preparation, manipulation, and measurement of quantum states and can be applied for quantum signal processing, quantum-state engineering, and quantum computing [81, 86–103]. Josephson charge qubit circuits for coherent quantum information processing are described in [97, 98]. Quantum superposition has the potential of high parallelism and efficiency, if the decoherence problem can be solved [104]. Using long arrays with 1000 Josephson junctions amplified in multiple eigenmodes with frequencies below 10 GHz has been achieved, and qubit quantum jumps were detected [105].

Quantum radar and metrology systems apply quantum phenomena to enhance measurement sensitivity. Due to the Heisenberg uncertainty principle, quantum mechanics imposes limits on the precision of radar measurements. The application of quantum mechanically entangled or squeezed light to illuminate objects can provide substantial enhancement of accuracy for detecting and imaging objects in the presence of high levels of noise and loss. Quantum radar technology is based on the phenomenon of quantum mechanical entanglement of states [106–111]. In quantum radar applications, Josephson-effect-based electronics can provide for squeezed-state generation or photon detection.

8. References

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