Exact Geometrical-Optics Scattering by a Class of Metallic Wedges Under Multiple Plane-Wave Illumination

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Abstract – Scattering by a metallic wedge with an aperture angle equal to π rad divided by an even integer is considered in the phasor domain. It is shown that under incidence by multiple plane waves of appropriate number, direction, phase, and polarization, the edge of the wedge does not scatter, and geometrical optics provides the exact, closed-form solution to the boundary-value problem.

1. Introduction

The exact scattering of a plane electromagnetic wave by a metallic (perfect electrical conductor [PEC]) wedge of arbitrary aperture angle was obtained by MacDonald [1] as an infinite series of circular-cylinder wave functions. Exact geometrical-optics solutions for wedge regions involving both metallic wedges and penetrable wedges either consisting of isorefractive material or subject to total transmission have been published [2–6]. The only known exact geometrical-optics solution to the scattering of a single plane wave by a wedge is that of a right-angle wedge made of double-negative metamaterial [7].

An exact geometrical-optics (GO) solution to the scattering of a single plane wave by a convex metallic wedge is not possible; however, such a solution exists for certain wedges if the primary field consists of more than one plane wave, as was recently shown for wedges with aperture angles of $\pi/2$ rad and $\pi/4$ rad [8, 9].

In this work, a metal wedge of aperture angle $\pi/2n$ rad is considered, where n is a positive integer. An exact GO solution is obtained in Section 2 when $4n-1$ plane waves with appropriate direction, polarization, and phase are incident upon the wedge. These results were recently presented at a conference [10].

2. Exact Geometrical-Optics Solution

Consider a PEC wedge with aperture angle $\pi/2n$ rad, where $n = 1, 2, 3, \ldots$ is a positive integer. With reference to circular-cylinder coordinates (ρ, φ, z) related to the rectangular coordinates (x, y, z) by $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, the edge of the wedge coincides with the z-axis and the faces of the wedge are the half-planes $\varphi = \pi$ and $\varphi = \pi + \pi/2n$. Starting from the positive x-axis ($\varphi = 0$), let us divide the space into $4n$ angular regions, each of angular width $\pi/2n$ rad; one region is occupied by the PEC wedge, and the other $4n-1$ regions by free space. There are $4n-1$ incident plane waves, incoming one in each free-space angular region, which are image-symmetric with respect to the boundaries of the angular regions. The directions of incidence of the primary plane waves are

$$\begin{aligned}\varphi_l^i &= (l-1)\frac{\pi}{2n} + \varphi_0 \quad (l \text{ odd} : l = 1, 3, \ldots, 4n-1),\\ &= l\frac{\pi}{2n} - \varphi_0 \quad (l \text{ even} : l = 2, 4, \ldots, 4n),\end{aligned} \tag{1}$$

where $0 \le \varphi_0 \le \pi/2n$ and $l = 2n+1$ is excluded because it corresponds to an image wave inside the PEC wedge.

For E-polarization (electric field everywhere parallel to the edge of the wedge), the incident waves have equal amplitude and are alternatively out of phase at the edge:

$$E_{lz}^i = (-1)^{l+1} e^{jk\rho\cos\left(\varphi-\varphi_l^i\right)}, \tag{2}$$

where $1 \le l \le 4n$ and $l = 2n+1$ is excluded. When the GO reflections of the incident waves on the faces of the wedge are added to the incident waves, the total field has no discontinuities across optical boundaries and consists of the sum of the $4n-1$ incident fields plus the field of the image wave inside the wedge:

$$\begin{aligned}E_z^{(e)} &= \sum_{l=1}^{4n} (-1)^{l+1} e^{jk\rho\cos\left(\varphi-\varphi_l^i\right)}\\ &= \sum_{m=1}^{2n} \left\{ e^{jk\rho\cos\left[\varphi-\varphi_0-(m-1)\frac{\pi}{n}\right]} - e^{jk\rho\cos\left(\varphi+\varphi_0-m\frac{\pi}{n}\right)} \right\};\end{aligned} \tag{3}$$

the corresponding magnetic-field components are obtained from (3) via Maxwell's equations:

$$\begin{aligned}H_\rho^{(e)} = Y \sum_{m=1}^{2n} \Big\{ &\sin\left[\varphi-\varphi_0-(m-1)\frac{\pi}{n}\right] e^{jk\rho\cos\left[\varphi-\varphi_0-(m-1)\frac{\pi}{n}\right]}\\ &- \sin\left[\varphi+\varphi_0-m\frac{\pi}{n}\right] e^{jk\rho\cos\left(\varphi+\varphi_0-m\frac{\pi}{n}\right)} \Big\},\end{aligned} \tag{4}$$

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$$H_\varphi^{(e)} = Y\sum_{m=1}^{2n}\left\{\cos\left[\varphi-\varphi_0-(m-1)\frac{\pi}{n}\right] e^{jk\rho\cos\left[\varphi-\varphi_0-(m-1)\frac{\pi}{n}\right]} - \cos\left[\varphi+\varphi_0-m\frac{\pi}{n}\right]e^{jk\rho\cos\left(\varphi+\varphi_0-m\frac{\pi}{n}\right)}\right\}, \quad (5)$$

where Y is the intrinsic admittance of free space. It follows from (4) that the surface current densities on the faces of the wedge are

$$J_z^{(e)}|_{\varphi=\pi} = Y\sum_{m=1}^{2n}\left\{\sin\left[\varphi_0+(m-1)\frac{\pi}{n}\right] e^{-jk\rho\cos\left[\varphi_0+(m-1)\frac{\pi}{n}\right]} + \sin\left(\varphi_0-m\frac{\pi}{n}\right)e^{-jk\rho\cos\left(\varphi_0-m\frac{\pi}{n}\right)}\right\}, \quad (6)$$

$$J_z^{(e)}|_{\varphi=\pi+\frac{\pi}{2n}} = -Y\sum_{m=1}^{2n}\left\{\sin\left[\varphi_0+\left(m-\frac{3}{2}\right)\frac{\pi}{n}\right] e^{-jk\rho\cos\left[\varphi_0+\left(m-\frac{3}{2}\right)\frac{\pi}{n}\right]} + \sin\left[\varphi_0-\left(m-\frac{1}{2}\right)\frac{\pi}{n}\right] e^{-jk\rho\cos\left[\varphi_0-\left(m-\frac{1}{2}\right)\frac{\pi}{n}\right]}\right\}. \quad (7)$$

It can be verified that the currents vanish at the edge $\rho = 0$ of the wedge.

For H-polarization (magnetic field everywhere parallel to the edge of the wedge), the incident waves have equal amplitude and are in phase at the edge of the wedge:

$$H_{lz}^i = Ye^{jk\rho\cos\left(\varphi-\varphi_l^i\right)}, \; (1 \le l \ne 2n+1 \le 4n) \quad (8)$$

and the total field is

$$H_z^{(h)} = Y\sum_{l=1}^{4n} e^{jk\rho\cos\left(\varphi-\varphi_l^i\right)} = Y\sum_{m=1}^{2n}\left\{e^{jk\rho\cos\left[\varphi-\varphi_0-(m-1)\frac{\pi}{n}\right]} + e^{jk\rho\cos\left(\varphi+\varphi_0-m\frac{\pi}{n}\right)}\right\}, \quad (9)$$

$$E_\rho^{(h)} = -\sum_{m=1}^{2n}\left\{\sin\left[\varphi-\varphi_0-(m-1)\frac{\pi}{n}\right] e^{jk\rho\cos\left[\varphi-\varphi_0-(m-1)\frac{\pi}{n}\right]} + \sin\left(\varphi+\varphi_0-m\frac{\pi}{n}\right) e^{jk\rho\cos\left(\varphi+\varphi_0-m\frac{\pi}{n}\right)}\right\}, \quad (10)$$

$$E_\varphi^{(h)} = -\sum_{m=1}^{2n}\left\{\cos\left[\varphi-\varphi_0-(m-1)\frac{\pi}{n}\right] e^{jk\rho\cos\left[\varphi-\varphi_0-(m-1)\frac{\pi}{n}\right]} + \cos\left(\varphi+\varphi_0-m\frac{\pi}{n}\right) e^{jk\rho\cos\left(\varphi+\varphi_0-m\frac{\pi}{n}\right)}\right\}. \quad (11)$$

The surface current densities are

$$J_\rho^{(h)}|_{\varphi=\pi} = -Y\sum_{m=1}^{2n}\left\{e^{-jk\rho\cos\left[\varphi_0+(m-1)\frac{\pi}{n}\right]} + e^{-jk\rho\cos\left(\varphi_0-m\frac{\pi}{n}\right)}\right\}, \quad (12)$$

$$J_\rho^{(h)}|_{\varphi=\pi+\frac{\pi}{2n}} = Y\sum_{m=1}^{2n}\left\{e^{-jk\rho\cos\left[\varphi_0+\left(m-\frac{2}{3}\right)\frac{\pi}{n}\right]} + e^{-jk\rho\cos\left[\varphi_0-\left(m-\frac{1}{2}\right)\frac{\pi}{n}\right]}\right\}. \quad (13)$$

The currents are perpendicular to the edge of the wedge and continuous across it.

It should be noted that the PEC boundary conditions are satisfied not only on the wedge surface but on all the half-planes $\varphi = (m-1)\pi/2n$, $m = 1, 2, \ldots 4n$; the technique used herein may be considered an extension of the method of images. The free space around the wedge is filled with $2n$ sets of plane waves traveling in opposite directions, thus forming $2n$ standing waves.

The solutions for the particular cases of wedges with aperture angles of $\pi/2$ rad ($n = 1$) and $\pi/4$ rad ($n = 2$) were presented in [8] and [9], respectively. For a wedge with aperture angle of $\pi/6$ rad ($n = 3$), it takes 11 incident plane waves to avoid scattering by the edge, and the total field components parallel to the edge are, from (3) and (9) with $n = 3$,

$$\left.\begin{array}{r} E_z^{(e)}|_{n=3} \\ Y^{-1}H_z^{(h)}|_{n=3} \end{array}\right\} = 2\left\{\cos[k\rho\cos(\varphi-\varphi_0)] \mp \cos[k\rho\cos(\varphi+\varphi_0)] + \cos\left[k\rho\cos\left(\varphi-\varphi_0-\frac{\pi}{3}\right)\right] \mp \cos\left[k\rho\cos\left(\varphi+\varphi_0-\frac{\pi}{3}\right)\right] + \cos\left[k\rho\cos\left(\varphi-\varphi_0+\frac{\pi}{3}\right)\right] \mp \cos\left[k\rho\cos\left(\varphi+\varphi_0+\frac{\pi}{3}\right)\right]\right\}, \quad (14)$$

where $0 \le \varphi_0 \le \pi/6$.

3. Discussion and Conclusion

A novel exact and closed-form canonical solution has been derived for the scattering of multiple plane waves by a class of metallic (PEC) wedges of aperture angle $\pi/2n$ rad, where n is a positive integer. This solution is possible because under the prescribed plane-wave illumination, the edge of the wedge does not scatter. Aside from the intrinsic value of a novel canonical solution of a boundary-value problem, the results obtained may be useful for the validation of computer solvers.

It is interesting to examine what happens to the solution when n tends to infinity—that is, when the aperture angle of the wedge tends to zero (half-plane case). The series (3) and (9) become integrals when the number of incident plane waves, each with infinitesimal amplitude, tends to infinity while φ_0 tends to zero. For E-polarization, the field is zero everywhere, which is a trivial result to be expected on the basis of physical considerations. For H-polarization, the incident field equals the total field and is given by

$$H_z^{(h)}|_{n\to\infty} = Y \int_{v=0}^{\pi} \cos[k\rho\cos(v-\varphi)]dv, \tag{15}$$

$$E_\rho^{(h)}|_{n\to\infty} = -j \int_{v=0}^{\pi} \sin[k\rho\cos(v-\varphi)]\sin(v-\varphi)dv, \tag{16}$$

$$E_\varphi^{(h)}|_{n\to\infty} = -j \int_{v=0}^{\pi} \sin[k\rho\cos(v-\varphi)]\cos(v-\varphi)dv. \tag{17}$$

It is easy to verify that (15) satisfies the scalar wave equation, that the tangential electric field (16) is zero on the faces $\varphi = \pi$ of the half-plane, and that the electric field is zero at the edge, meaning that the edge does not scatter.

Starting from the exact and closed-form solution of the scattering of a plane wave by a metallic half-plane [11] and utilizing a property of Fresnel integrals, it can be seen that the edge of the half-plane does not scatter for either polarization when it is illuminated by two plane waves of appropriate phase and polarization that are incident on the edge symmetrically with respect to the half-plane [10]. Therefore, the solution (15–17) is not the only field configuration for which the edge of the half-plane does not scatter.

Consequently, it can be stated that the set of incident plane waves described in this work constitutes a sufficient but not a necessary condition to avoid scattering by the edge of the wedge and obtain an exact GO solution.

A general discussion of the conditions under which geometrical optics may yield an exact solution is found in [12]. In order to avoid scattering by the edge of a wedge, it is not sufficient to eliminate field singularities near the edge, a result that can be achieved for a PEC wedge of arbitrary aperture angle under incidence by two plane waves [13, section 4.4]; it is also necessary that the boundary conditions on the wedge surface be satisfied, and that no field discontinuities occur across optical boundaries. Finally, the results obtained in this work can be extended easily to the case of a perfect magnetic conductor (PMC) wedge by applying the duality principle.

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