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Exact analytical approach to TM-wave oblique incidence on impedance-matched graded interfaces between RHM and LHM media

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Abstract

We investigate TM-wave propagation over an impedance-matched graded interface between a right-handed material (RHM) and a left-handed material (LHM) with graded permittivity and permeability. We assume a lossy case with permittivity and permeability profile changing according to a hyperbolic tangent function along the direction perpendicular to the boundary plane between the two materials. A remarkably simple exact analytical solution to Helmholtz equation is obtained. The exact analytical expression for the field intensity along the graded LHM-RHM structure confirms the expected properties of LHM media. Furthermore, in the special case of normal incidence, we reconfirm that there is an excellent agreement between the present analytical results and numerical simulations.

1 Introduction

There is a growing theoretical and practical interest for left-handed metamaterials with spatially varying permittivities and permeabilities within the elctromagnetics community. Such graded permittivities and permeabilities are utilized in a number of research areas. One major area of interest is transformation optics [1] with hyperlenses [2, 3], antennas [4, 5] and subwavelength imaging [6, 7]. Another area of interest is waveguide applications [8, 9], in particular nanostructured waveguides proposed to enhance the performance of solar cells via a tunable absorption spectrum [10]. Yet another area is invisibility cloaks [11]. In a short paper of the present format, it is not possible to give an extensive account of all applications of graded left-handed metamaterials, or to list a large number of references. The references cited are therefore just a few representative examples.

In the present paper, we study TM-wave propagation over an impedance matched graded interface between a right-handed material (RHM) and a left-handed material (LHM). An early similar study for the lossless case of TE-wave propagation over an impedance matched graded interface between a right-handed material (RHM) and a left-handed material (LHM), was reported in [12]. For the most general analytical approach to a non-impedance-matched graded interface between a right-handed material (RHM) and a left-handed material (LHM), see [13]. The exact analytical solution in [12] was mathematically complex and obtained in terms of Gaussian hypergeometric functions. Furthermore, the field patterns were not investigated in detail to clearly relate to the important properties of left-handed metamaterials. Using a combination of properties of Gaussian hypergeometric functions [15], in the present paper we obtain an analogous mathematical solution. However, the solution here is generalized to a lossy case and TM-waves, and involves elementary mathematical functions only. Thus we obtain a remarkably simple exact analytical solution to Helmholtz' equation for a lossy case with permittivity and permeability profile, changing according to a hyperbolic tangent function along the direction perpendicular to the boundary plane between the two materials. Furthermore, we provide a three-dimensional graphical presentation and discussion of the results.

2 Field equations with solutions

We assume that the material can be described by its effective dielectric permittivity and effective magnetic permeability, which is normally justified for left-handed metamaterials, since their 'particles' have subwavelength dimensions. We choose the geometry of the problem such that the effective parameters change along one direction only (chosen to be the x-axis). The plane of incidence is chosen to be the $x-y$ plane, such that the direction perpendicular to the plane of incidence is the z-direction. For a TM-wave, for an oblique incidence at angle θ, the magnetic and electric field vectors are then given respectively by

$$\boldsymbol{H}(x,y) = H(x,y)\,\hat{\boldsymbol{z}} \tag{1}$$

$$\boldsymbol{E}(x,y) = -E(x,y)\sin\theta\,\hat{\boldsymbol{x}} + E(x,y)\cos\theta\,\hat{\boldsymbol{y}} \tag{2}$$

and the incident wave vector is $\boldsymbol{k} = k\cos\theta\,\hat{\boldsymbol{x}} + k\sin\theta\,\hat{\boldsymbol{y}}$. The Helmholtz equation for the magnetic field intensity is then obtained in the form

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} - \frac{1}{\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}x}\frac{\partial H}{\partial x} + \omega^2\varepsilon\mu H = 0 \tag{3}$$

For the lossy impedance-matched graded interface between a right-handed material (RHM) and a left-handed material (LHM), it is convenient to define the permittivity and permeability functions, changing according to a hyperbolic tangent function along the x-direction, as follows

$$\varepsilon(\omega,x) = -\varepsilon_0 \varepsilon_R(\omega)\left[\tanh\left(\frac{x}{x_0}\right) + \mathrm{i}\beta\right] \tag{4}$$

$$\mu(\omega,x) = -\mu_0 \mu_R(\omega)\left[\tanh\left(\frac{x}{x_0}\right) + \mathrm{i}\beta\right] \tag{5}$$

where x_0 is the length parameter that defines the width of the graded transition region between the two media. The loss parameter β is the ratio between the imaginary and real parts of the permittivity and permeability functions (4) and (5). This parameter is assumed to be equal, and to a good approximation constant, for the chosen dispersion model in the frequency range of interest. Separating the variables using $H(x,y) = X(x)Y(y)$ as in [12], it is easily shown that the solution for $Y(y)$ is a simple attenuated plane wave. The ordinary differential equation for $X(x)$ can be reduced to the hypergeometric differential equation, with a solution proportional to a Gaussian hypergeometric function ${}_2F_1(a,b,c;u)$. However, in present study, using some properties of Gaussian hypergeometric functions [15], we have reduced the solution for $X(x)$ to a form involving elementary mathematical functions only. The exact analytical solution has the following simple form

$$\begin{aligned} H(x,y) &= H_0\ e^{-\beta k(x\cos\theta + y\sin\theta)}\ \cdot \\ &\cdot \left[\cosh\left(\frac{x}{x_0}\right)\right]^{\mathrm{i}kx_0\cos\theta} e^{-\mathrm{i}ky\sin\theta} \end{aligned} \tag{6}$$

where we introduce the notation H_0 for the magnetic field strength at the origin $(x,y) = (0,0)$. The solution for the electric field strength $E(x,y)$ is proportional to the solution for the magnetic field strength $H(x,y)$, and has a fully analogous mathematical form

$$\begin{aligned} E(x,y) &= E_0\ e^{-\beta k(x\cos\theta + y\sin\theta)}\ \cdot \\ &\cdot \left[\cosh\left(\frac{x}{x_0}\right)\right]^{\mathrm{i}kx_0\cos\theta} e^{-\mathrm{i}ky\sin\theta} \end{aligned} \tag{7}$$

where E_0 is the electric field strength at $(x,y) = (0,0)$.

3 Graphical presentation and discussion

We present the electric field intensity pattern $E(x,y)$ defined by (7) for a TM-wave in the microwave range of frequencies with the normalization amplitude at the origin $E_0 = 1$, the wave number k such that $kx_0 = 0.2$, the loss factor $\beta = 0.05$ and angle of incidence (a) $\theta = \pi/6$ and (b) $\theta = \pi/3$, in Figure 1. The electric field wave pattern, for the same set of numerical parameters as in Figure 1, can be found in Figure 2. The cross section of the electric field pattern shown in Figure 2 for constant $y = \pi/2$ is shown in Figure 3, with otherwise the same parameters as the previous two figures. The function shown in Figure 3 describes the wave propagation along the x-axis only.

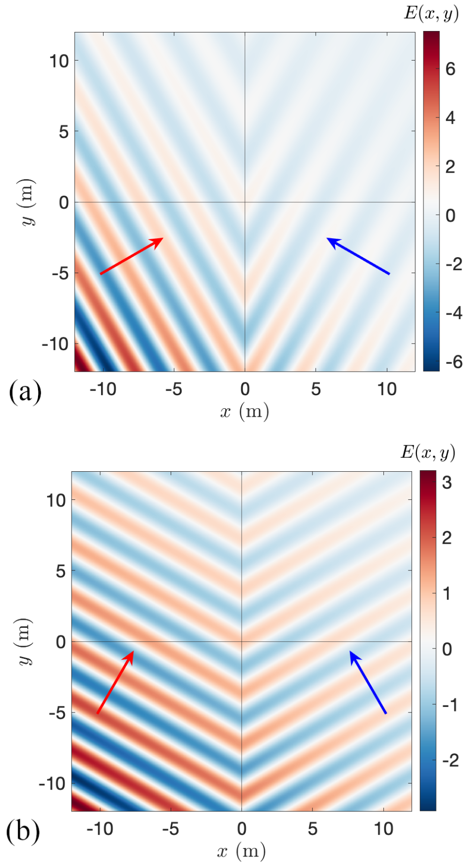

Figure 1. Electric field intensity pattern for $E_0 = 1$, $kx_0 = 0.2$ and $\beta = 0.05$, with angles of incidence (a) $\theta = \pi/6$ and (b) $\theta = \pi/3$. Arrows indicate wave vector directions for RHM (red) and LHM (blue).

For normal incidence, when $\theta = 0$, in the lossless case with $\beta = 0$, the result (7) is reduced to the previously obtained result in [14]

$$E(x) = E_0\left[\cosh\left(\frac{x}{x_0}\right)\right]^{\mathrm{i}kx_0} \tag{8}$$

From Figure 3. in [14], we know that the result (8) was validated by comparison of the exact analytical wave forms with the corresponding wave forms obtained by numerical simulation of Maxwell's equations using the finite element method (COMSOL Multiphysics). In [14], an excellent agreement between the analytical and numerical results was obtained. Since the results in [14] are just one special case of the present more general results, the same level of agreement with numerical simulations is expected for the results obtained in the present paper.

In the impedance-matched case, as the one studied here and in [12] and [14], we know that there is no reflection at the interface between the two media. In the case of normal incidence, described by (8), it means that the wave seemingly continues to propagate undisturbed over the boundary between the two media. However, one important physical observation is the reversed sign of the wave vector in the left-

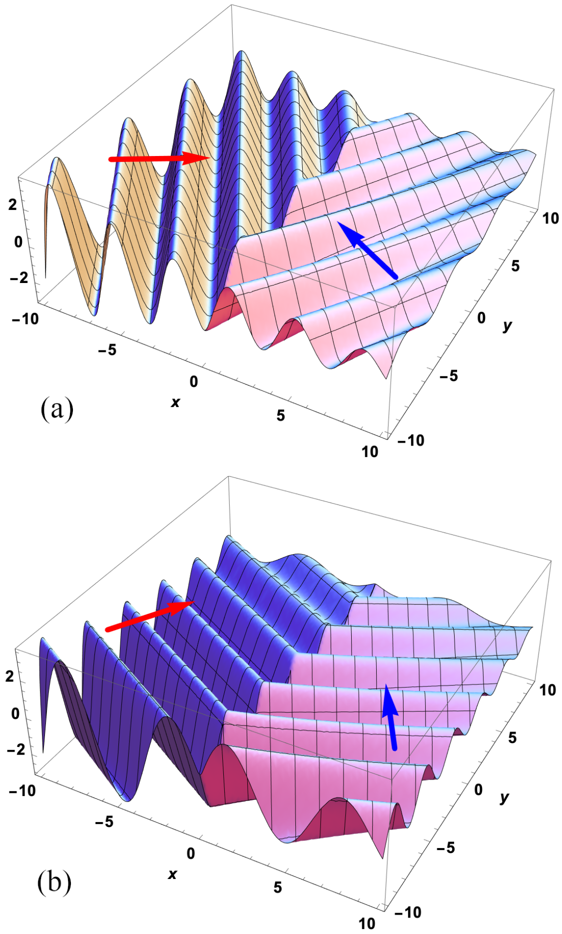

Figure 2. Electric field wave pattern for $E_0 = 1$, $kx_0 = 0.2$ and $\beta = 0.05$, with angles of incidence (a) $\theta = \pi/6$ and (b) $\theta = \pi/3$. Arrows indicate wave vector directions for RHM (red) and LHM (blue).

handed material, being a well-known property of LHM media. Thus, in the special case of normal incidence, the wave continues to propagate over the boundary between two very different media, and only changes sign of the wave vector (direction of the wave vector along the same straight line).

On the other hand, for oblique incidence, the wave vector $\boldsymbol{k} = k\cos\theta\,\hat{\boldsymbol{x}} + k\sin\theta\,\hat{\boldsymbol{y}}$ has both x- and y-components. And since the media change from RHM to LHM occurs only in the x-direction, only the sign of the x-component of the wave vector is changed, while the sign of the y-component is unchanged. Thus in Figures 1 and 2, we see that despite the impedance-matching and no wave reflection at the boundary between the two materials, the incident TM-wave does not continue along the same path in the LHM medium.

Wave refraction occurs at the boundary between the two materials, and the wave not only propagates in the opposite direction from what is expected in RHM media, it also propagates along a path that is a mirror image of the incident wave. This observation can be interpreted as an RHM-LHM interface version of Snell's law of refraction

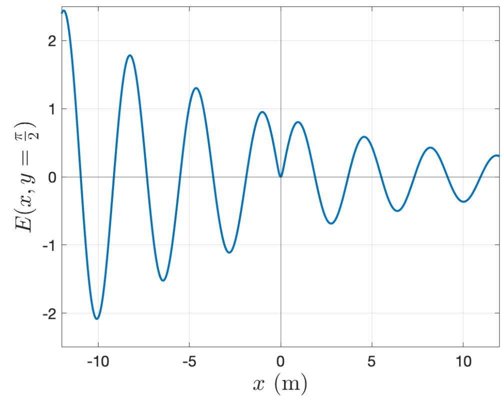

Figure 3. Electric field pattern along $y = \pi/2$ with $E_0 = 1$, $kx_0 = 0.2$, $\beta = 0.05$ and $\theta = \pi/6$.

$n_t \sin\theta_t = n_i \sin\theta_i$, where $n_t = -n_i$ is the negative refractive index of the left-handed material, while n_i is the positive refractive index of the right-handed material. Thus, $\sin\theta_t = -\sin\theta_i$ which implies $\theta_t = -\theta_i$, in agreement with the behavior established in the present analysis.

It should be noted however that in the present treatment, we do not need to use any law of refraction, or any boundary conditions. The solution of Maxwell's equations for a single stratified medium reproduces the correct LHM media behavior without any a priori assumptions. The present approach also includes also the abrupt transition, i.e. a sharp interface between the two materials, as a special case when the width of transition region x_0 approaches zero ($x_0 \to 0$). This peculiar property of LHM media may have implications for some invisibility cloaking designs.

Finally, far from the boundary surface between the two media, the result (7) with $\beta = 0$, reproduces the expected asymptotic plane-wave solutions for $x \to -\infty$

$$E(x,y) = E_0 e^{-\mathrm{i}\alpha} e^{-\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{x}} = E_0 e^{-\mathrm{i}\alpha} e^{-\mathrm{i}(kx\cos\theta + ky\sin\theta)} \quad (9)$$

and for $x \to +\infty$

$$E(x,y) = E_0 e^{-\mathrm{i}\alpha} e^{-\mathrm{i}\boldsymbol{k}_t\cdot\boldsymbol{x}} = E_0 e^{-\mathrm{i}\alpha} e^{-\mathrm{i}(-kx\cos\theta + ky\sin\theta)} \quad (10)$$

where $\boldsymbol{k}_i$ and $\boldsymbol{k}_t$ are complex wave vectors in RHM and LHM media, respectively, and $\alpha = kx_0 \ln 2\cos\theta$ is an unessential constant phase shift that can be included in the complex amplitude E_0. These asymptotic results reconfirm that the directions of two wave vectors in the two media are indeed in agreement with the wave behavior displayed in Figures 1 and 2.

4 Conclusions

We studied TM-wave propagation over impedance-matched interfaces between lossy RHM and LHM media with spatially graded permittivities and permeabilities. The spatial gradient is described using a hyperbolic tangent function along the direction perpendicular to the boundary plane between the two materials. We derived remarkably simple exact analytical solutions to the Helmholtz equation. The obtained exact analytical expressions for the field intensities along the graded LHM-RHM structure confirm all the expected properties of LHM media, and are in agreement with the results of previous studies. In the special case of normal incidence without losses, the present study reconfirms an excellent agreement between the analytical results and accurate numerical simulations.

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