# Scattering of an Oblique Electromagnetic Wave by a Metal Cylinder in a Cold Resonant Magnetoplasma 

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#### Abstract

The exact analytical solution to the problem of scattering of an oblique electromagnetic wave by a metal cylinder in a cold resonant magnetoplasma is found. It is shown that a quasi-electrostatic wave, which wavelength is much smaller than the radius of cylinder, is scattered mostly in the forward direction in terms of energy.


## 1 Introduction

A problem of scattering of electromagnetic waves by metal objects in magnetoplasmas has a long history [1-4]. In recent decades, interest in this problem has increased due to detection of oblique waves onboard spacecraft in the nearEarth plasma: the correct calculation of the wave electric field values from the voltage data is sometimes challenging because of the intense re-radiation [5-7].

In this paper, the exact analytical solution to the problem of scattering of an oblique electromagnetic wave by a metal cylinder parallel to the ambient magnetic field in a cold resonant magnetoplasma is found. It is valid for an arbitrary (but permissible by the plasma dispersion properties) wave normal angle. Much attention is paid to the case when this angle is close to the resonance cone angle.

## 2 Geometry of the Problem and Plasma Dispersion Properties

We consider a plane monochromatic wave scattered by a perfectly conducting circular infinitely long cylinder of radius $a$ parallel to the ambient magnetic field in a cold collisionless magnetoplasma (see Figure 1) with the dielectric tensor

$$
\hat{\boldsymbol{\varepsilon}}=\left(\begin{array}{ccc}
\varepsilon & -i g & 0  \tag{1}\\
i g & \varepsilon & 0 \\
0 & 0 & \eta
\end{array}\right)
$$

where

$$
\varepsilon=1+\frac{\omega_{\mathrm{p}}^{2}}{\omega_{\mathrm{c}}^{2}-\omega^{2}}, \quad g=-\frac{\omega_{\mathrm{p}}^{2} \omega_{\mathrm{c}}}{\left(\omega_{\mathrm{c}}^{2}-\omega^{2}\right) \omega}, \quad \eta=1-\frac{\omega_{\mathrm{p}}^{2}}{\omega^{2}} .
$$

Here, $\omega_{\mathrm{c}}$ and $\omega_{\mathrm{p}}$ are the electron cyclotron and plasma frequencies, respectively. In what follows, we limit ourselves
to the resonant frequency range

$$
\begin{equation*}
\omega_{\mathrm{c}} / 2<\omega<\omega_{\mathrm{c}} \ll \omega_{\mathrm{p}} \tag{2}
\end{equation*}
$$

In this range, $\varepsilon>0, \eta<0$, and the corresponding wave normal surface is confined to the resonance cone direction and does not include inflection points (see Figure 2).

## 3 Incident Wave

The incident wave is described by the angle $\theta_{\mathrm{i}}$ between $z$ axis and its wavevector $\mathbf{k}_{\mathrm{i}}$ that belongs to $(x, z)$-plane (see Figure 1). This angle specifies the phase propagation direction for the incident wave and, because of the dispersion properties in range (2), can be varied in range $0 \leq$ $\theta_{i}<\theta_{\text {res }} \equiv \arctan \sqrt{|\eta / \varepsilon|}$. The corresponding wavenumber $k_{\mathrm{i}} \equiv\left|\mathbf{k}_{\mathrm{i}}\right|$ is determined from $\theta_{\mathrm{i}}$ using the dispersion relation:

$$
\begin{equation*}
\left(\frac{k_{\mathrm{i}}}{k_{0}}\right)^{2}=\frac{2 \varepsilon \eta+\left(\varepsilon^{2}-g^{2}-\varepsilon \eta\right) \sin ^{2} \theta_{\mathrm{i}}-D^{1 / 2}}{2\left(\varepsilon \sin ^{2} \theta_{\mathrm{i}}+\eta \cos ^{2} \theta_{\mathrm{i}}\right)} \tag{3}
\end{equation*}
$$

where $D=\left(\varepsilon^{2}-g^{2}-\varepsilon \eta\right)^{2} \sin ^{4} \theta_{\mathrm{i}}+4 g^{2} \eta^{2} \cos ^{2} \theta_{\mathrm{i}}$ and $k_{0}=\omega / c$.

The electric and magnetic fields of the incident wave, with the time factor $\exp (i \omega t)$ dropped, are

$$
\left[\begin{array}{l}
\mathbf{E}_{\mathrm{i}}  \tag{4}\\
\mathbf{H}_{\mathrm{i}}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{e}_{\mathrm{i}} \\
\mathbf{h}_{\mathrm{i}}
\end{array}\right] \exp \left[-i k_{0}\left(q_{\mathrm{i}} x+p_{\mathrm{i}} z\right)\right],
$$

where $q_{\mathrm{i}}=\left(k_{\mathrm{i}} / k_{0}\right) \sin \theta_{\mathrm{i}}$ and $p_{\mathrm{i}}=\left(k_{\mathrm{i}} / k_{0}\right) \cos \theta_{\mathrm{i}}$. From the wave polarization coefficients, we obtain

$$
\begin{gather*}
e_{\mathrm{i} y}=\frac{i g}{q_{\mathrm{i}}^{2}+p_{\mathrm{i}}^{2}-\varepsilon} e_{\mathrm{i} x}, \quad e_{\mathrm{i} z}=\frac{q_{\mathrm{i}} p_{\mathrm{i}}}{q_{\mathrm{i}}^{2}-\eta} e_{\mathrm{i} x}, \\
h_{\mathrm{i} x}=-\frac{i g p_{\mathrm{i}}}{q_{\mathrm{i}}^{2}+p_{\mathrm{i}}^{2}-\varepsilon} \frac{e_{\mathrm{i} x}}{Z_{0}}, \quad h_{\mathrm{i} y}=-\frac{\eta p_{\mathrm{i}}}{q_{\mathrm{i}}^{2}-\eta} \frac{e_{\mathrm{i} x}}{Z_{0}}, \\
h_{\mathrm{i} z}=\frac{i g q_{\mathrm{i}}}{q_{\mathrm{i}}^{2}+p_{\mathrm{i}}^{2}-\varepsilon} \frac{e_{\mathrm{i} x}}{Z_{0}} \tag{5}
\end{gather*}
$$

where $Z_{0}$ is the impedance of free space. Note that if $\theta_{\mathrm{i}} \rightarrow \theta_{\text {res }}$, the magnetic field components tend to zero. In what follows, we normalize the incident wave as

$$
\begin{equation*}
\left|e_{i x}\right|^{2}+\left|e_{i y}\right|^{2}+\left|e_{i z}\right|^{2}=1 \tag{6}
\end{equation*}
$$



Figure 1. Geometry of the problem.

## 4 General Solution to Maxwell's Equations in a Magnetoplasma in Cylindrical Coordinates

Because of the symmetry of the problem (see Figure 1), the electric and magnetic fields of the scattered wave can be represented as

$$
\left[\begin{array}{c}
\mathbf{E}_{\mathrm{s}}  \tag{7}\\
\mathbf{H}_{\mathrm{s}}
\end{array}\right]=\sum_{m=-\infty}^{+\infty}\left[\begin{array}{c}
\mathscr{E}_{m}(\rho) \\
\mathscr{H}_{m}(\rho)
\end{array}\right] \exp \left(-i m \varphi-i k_{0} p z\right)
$$

The expressions for the components of $\mathscr{E}_{m}$ and $\mathscr{H}_{m}$ can be found analytically from Maxwell's equations in a magnetoplasma with tensor (1) (for details, see [8]):

$$
\begin{align*}
& \mathscr{E}_{m, \rho}(\rho)=-\sum_{k=1}^{2} A_{k, m}\left[\frac{n_{k} p+g}{\varepsilon} H_{m+1}^{(1)}\left(k_{0} q_{k} \rho\right)\right. \\
&\left.+q_{k} \frac{n_{k} p+\eta}{\eta\left(p^{2}-\varepsilon+g\right)} \frac{m}{k_{0} \rho} H_{m}^{(1)}\left(k_{0} q_{k} \rho\right)\right],  \tag{8}\\
& \mathscr{E}_{m, \varphi}(\rho)=i \sum_{k=1}^{2} A_{k, m}\left[H_{m+1}^{(1)}\left(k_{0} q_{k} \rho\right)\right. \\
&\left.+q_{k} \frac{n_{k} p+\eta}{\eta\left(p^{2}-\varepsilon+g\right)} \frac{m}{k_{0} \rho} H_{m}^{(1)}\left(k_{0} q_{k} \rho\right)\right]  \tag{9}\\
& \mathscr{E}_{m, z}(\rho)=\frac{i}{\eta} \sum_{k=1}^{2} A_{k, m} n_{k} q_{k} H_{m}^{(1)}\left(k_{0} q_{k} \rho\right),  \tag{10}\\
& \mathscr{H}_{m, \rho}(\rho)=-\frac{i}{Z_{0}} \sum_{k=1}^{2} A_{k, m}\left[p H_{m+1}^{(1)}\left(k_{0} q_{k} \rho\right)\right. \\
&\left.+q_{k} \frac{p \eta+n_{k}(\varepsilon-g)}{\eta\left(p^{2}-\varepsilon+g\right)} \frac{m}{k_{0} \rho} H_{m}^{(1)}\left(k_{0} q_{k} \rho\right)\right]  \tag{11}\\
& \mathscr{H}_{m, \varphi}(\rho)=-\frac{1}{Z_{0} \sum_{k=1}^{2} A_{k, m}\left[n_{k} H_{m+1}^{(1)}\left(k_{0} q_{k} \rho\right)\right.} \\
&\left.+q_{k} \frac{p \eta+n_{k}(\varepsilon-g)}{\eta\left(p^{2}-\varepsilon+g\right)} \frac{m}{k_{0} \rho} H_{m}^{(1)}\left(k_{0} q_{k} \rho\right)\right]  \tag{12}\\
& \mathscr{H} m, z(\rho)=-\frac{1}{Z_{0}} \sum_{k=1}^{2} A_{k, m} q_{k} H_{m}^{(1)}\left(k_{0} q_{k} \rho\right) \tag{13}
\end{align*}
$$

where $H_{m}^{(1)}(\cdot)$ is Hankel function of the first kind,

$$
\begin{equation*}
n_{k}=-\frac{\varepsilon}{p g}\left(p^{2}+q_{k}^{2}+\frac{g^{2}}{\varepsilon}-\varepsilon\right) \tag{14}
\end{equation*}
$$

$$
\begin{align*}
& q_{k}^{2}=q_{k}^{2}(p) \\
& \quad=\frac{1}{2 \varepsilon}\left[\varepsilon^{2}-g^{2}+\varepsilon \eta-(\eta+\varepsilon) p^{2}+(-1)^{k} R(p)\right] \tag{15}
\end{align*}
$$



Figure 2. The wave normal surface in frequency range (2) and the corresponding group $\left(\mathbf{V}_{\mathrm{gr}}\right)$ and phase $\left(\mathbf{V}_{\mathrm{ph}}\right)$ velocities. Quantities $p$ and $q$ are the longitudinal and transverse components of the wave normal vector, respectively.

$$
\begin{array}{r}
R(p)=\left\{(\eta-\varepsilon)^{2} p^{4}+2\left[g^{2}(\eta+\varepsilon)-\varepsilon(\eta-\varepsilon)^{2}\right] p^{2}\right. \\
\left.+\left(\varepsilon^{2}-g^{2}-\varepsilon \eta\right)^{2}\right\}^{1 / 2} \tag{16}
\end{array}
$$

Here we assume that $\mathfrak{R}[R(p)]>0$ and $\mathfrak{I}\left[q_{k}(p)\right]>0$. The latter inequality corresponds to the vanishingly small fields at $\rho \rightarrow+\infty$. If $\mathfrak{I}\left[q_{k}(p)\right]=0$, then the weak collisional dissipation should be introduced and its rate should be tended to zero.

In (8)-(13), the terms with $k=1$ correspond to the evanescent O mode, and $q_{1}$ is purely imaginary. The terms with $k=2$ correspond to the propagating X mode, and $q_{2}>0$.

Hankel function of the first kind corresponds to the ingoing waves of phase and therefore to the outgoing waves of energy in plane $z=$ const (see Figures 1 and 2). This is in agreement with the radiation conditions at infinity in anisotropic media [4].

## 5 Partial Solution to the Scattering Problem

A solution to the scattering problem-namely, the propagation constant $p$ and coefficients $A_{1, m}$ and $A_{2, m}$-follows from the boundary conditions for a perfect electrical conductor (see Figure 1):

$$
\begin{equation*}
\left.\left(E_{\mathrm{s} \varphi}+E_{\mathrm{i} \varphi}\right)\right|_{\rho=a}=0,\left.\quad\left(E_{\mathrm{s} z}+E_{\mathrm{i} z}\right)\right|_{\rho=a}=0 . \tag{17}
\end{equation*}
$$

From that we have $p=p_{\mathrm{i}}$ and

$$
\begin{gather*}
A_{1, m}=-\frac{\tilde{\Phi}_{m} \tilde{Z}_{2, m}-\tilde{\Phi}_{2, m} \tilde{Z}_{m}}{\tilde{\Phi}_{2, m} \tilde{Z}_{1, m}-\tilde{\Phi}_{1, m} \tilde{Z}_{2, m}}  \tag{18}\\
A_{2, m}=\frac{\tilde{\Phi}_{m} \tilde{Z}_{1, m}-\tilde{\Phi}_{1, m} \tilde{Z}_{m}}{\tilde{\Phi}_{2, m} \tilde{Z}_{1, m}-\tilde{\Phi}_{1, m} \tilde{Z}_{2, m}} \tag{19}
\end{gather*}
$$

where


Figure 3. Dependence of $k_{\mathrm{i}} a \sin \theta_{\mathrm{i}}$ on $\theta_{\mathrm{i}}$ for $0<\theta_{\mathrm{i}}<\theta_{\mathrm{res}}$, $a=$ const, and $k_{\mathrm{i}}=k_{\mathrm{i}}\left(\theta_{\mathrm{i}}\right)$.

$$
\begin{gathered}
\tilde{\Phi}_{k, m}=i H_{m+1}^{(1)}\left(k_{0} q_{k} a\right)+\frac{i q_{k}\left(n_{k} p_{\mathrm{i}}+\eta\right)}{\eta\left(p_{\mathrm{i}}^{2}-\varepsilon+g\right)} \frac{m}{k_{0} a} H_{m}^{(1)}\left(k_{0} q_{k} a\right), \\
\tilde{\Phi}_{m}=-e_{\mathrm{i} x} \frac{m(-i)^{m}}{k_{0} q_{\mathrm{i}} a} J_{m}\left(k_{0} q_{\mathrm{i}} a\right) \\
\quad-\frac{(-i)^{m+1}}{2} e_{\mathrm{i} y}\left[J_{m+1}\left(k_{0} q_{\mathrm{i}} a\right)-J_{m-1}\left(k_{0} q_{\mathrm{i}} a\right)\right], \\
\tilde{Z}_{k, m}=\frac{i}{\eta} n_{k} q_{k} H_{m}^{(1)}\left(k_{0} q_{k} a\right), \quad \tilde{Z}_{m}=-(-i)^{m} e_{\mathrm{i} z} J_{m}\left(k_{0} q_{\mathrm{i}} a\right)
\end{gathered}
$$

for $k=1,2$. Here, $J_{m}(\cdot)$ is Bessel function of the first kind.
In (8)-(13), the terms with $k=1$ vanish at some distance from the cylinder but contribute to the field structure in its vicinity. At $\rho \rightarrow+\infty$, only the terms with $k=2$ survive. Furthermore, in (8), (9), (11), and (12), only the first terms in square brackets prevail at $\rho \rightarrow+\infty$ for $k=2$. Consequently, the radar cross-section per unit length equals

$$
\begin{equation*}
\sigma(\varphi)=\left|\sum_{m=-\infty}^{+\infty} A_{2, m} \exp \left(-\frac{i \pi m}{2}-i m \varphi\right)\right|^{2} \tag{20}
\end{equation*}
$$

## 6 Calculation Results

The scattering characteristics were calculated for the plasma parameters typical for the Earth's magnetosphere (namely, at the geomagnetic equator for McIlwain parameter $L=5$ ): $\omega_{\mathrm{p}} \approx 1.8 \times 10^{5} \mathrm{~s}^{-1}$ (plasma density $N_{\mathrm{e}}=$ $10 \mathrm{~cm}^{-3}$ ), $\omega_{\mathrm{c}} \approx 3.8 \times 10^{4} \mathrm{~s}^{-1}$, and for the radiation frequency $\omega \approx 2.3 \times 10^{4} \mathrm{~s}^{-1}$. The corresponding dependence of parameter $\xi \equiv k_{\mathrm{i}} a \sin \theta_{\mathrm{i}}$ on $\theta_{\mathrm{i}}$ for $0<\theta_{\mathrm{i}}<\theta_{\mathrm{res}}, a=$ const, and $k_{\mathrm{i}}=k_{\mathrm{i}}\left(\theta_{\mathrm{i}}\right)$ [see (3)] is shown in Figure 3. When $\theta_{\mathrm{i}} \rightarrow \theta_{\mathrm{res}}$, wavenumber $k_{\mathrm{i}}$ and therefore parameter $\xi$ increase significantly due to the nature of quasi-electrostatic waves.

The radar cross-section per unit length $\sigma(\varphi)$ in plane $z=$ const for different values of $\theta_{\mathrm{i}} / \theta_{\text {res }}$ is shown in Figure 4.


Figure 4. The radar cross-section per unit length $\sigma(\varphi)$ in plane $z=$ const for different values of $\theta_{\mathrm{i}} / \theta_{\text {res }}$. Energy in the incident wave propagates from the right side (from $\varphi=0^{\circ}$ to $\varphi=180^{\circ}$ ).

When $\xi \ll 1$, scattering is quasi-isotropic. When $\theta_{\mathrm{i}} \rightarrow \theta_{\text {res }}$ and $\xi \gtrsim 1$, scattering takes place predominantly in the direction $\varphi=180^{\circ}$-the same direction as the incident wave energy propagates in (see Figure 1). In terms of energy, this is the forward scattering process.

## 7 Conclusion

The results are in agreement with the general scattering theory [9]: small (such that $k_{\mathrm{i}} a \sin \theta_{\mathrm{i}} \ll 1$ ) scatterers are characterized by isotropic scattering, whereas forward scattering corresponds to large scatterers.

The problem considered in this paper corresponds to quite a simple model: the Debye sheath around the cylinder and collisions in the plasma are neglected. However, these effects can be taken into account using the same approach.

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