# Ray-tube integration in shooting and bouncing ray method revisited 

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#### Abstract

The method of shooting and bouncing rays (SBR) was originally developed for radar cross section (RCS) analysis of aircraft engines, but is today widely used in many applications requiring high-frequency approximations. A dense grid of ray-tubes are launched from the source and traced by the laws of geometrical optics (GO) until they reach a defined exit aperture, from which the far field contribution of the ray-tube is obtained by surface integration. Two alternatives have been suggested on where to perform the surface integration, namely (1) on the ray-tube cross-section or (2) on the exit aperture. The first option is convenient as the electrical field is assumed constant in the ray-tube surface and has been suggested for complex scattering problems. In this paper, it is demonstrated that the surface integration over the ray-tube area can cause a ripple anomaly. It is illustrated that the ripple is caused by omitting parts of the integration surface from which there are no power flow. It is concluded that exit aperture integration should be selected in SBR analysis.


## 1 Introduction

The shooting and bouncing ray (SBR) method was originally developed for radar cross-section analysis of aircraft engine cavities [1], but was later extended for dielectrics [2] and has recently been found useful for the analysis of radomes [3]. In SBR, a large number of rays are launched to hit the scattering object (e.g. the aircraft cavity). The rays are then traced using the laws of geometrical optics (GO), until the ray reaches a defined exit aperture, from which the far field contribution of the ray is obtained by surface integration.

Two alternatives are suggested on where to perform the surface integration [4], namely: (1) over the ray-tube area or (2) over the exit aperture. The first option is convenient as the electrical field is assumed constant in the ray-tube surface and has been suggested for complex scattering problems [4]. In this paper, it is demonstrated that the surface integration over the ray-tube area can cause a ripple anomaly. By demonstration for simple geometries, it is illustrated that the ripple is caused by omitting parts of the integration surface from which there are no power flow.


Figure 1. The sources are contained inside the volume $V$.

In the next section, the integral representations which are the foundation of the surface integration in SBR are briefly revisited. The ripple anomaly is then demonstrated for simple spherical geometries in the following sections.

## 2 Method

### 2.1 Integral representations

The integral representations for the electromagnetic fields (see e.g. [5]) show that the time-harmonic electromagnetic fields (with time dependence $e^{-i \omega t}$ ) in an arbitrary observation point $r$ outside the source volume $V$ can be represented by surface integrals of the tangential electric $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ over the boundary of $V$ as

$$
\begin{align*}
i \frac{\eta_{0}}{k} \nabla \times & {\left[\nabla \times \iint_{\partial V} g\left(k,\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|\right)\left(\hat{\boldsymbol{n}}\left(\boldsymbol{r}^{\prime}\right) \times \boldsymbol{H}\left(\boldsymbol{r}^{\prime}\right)\right) d S^{\prime}\right] } \\
+ & \nabla \times \iint_{\partial V} g\left(k, \boldsymbol{r}-\boldsymbol{r}^{\prime}\right)\left(\hat{\boldsymbol{n}}\left(\boldsymbol{r}^{\prime}\right) \times \boldsymbol{E}\left(\boldsymbol{r}^{\prime}\right)\right) d S^{\prime}  \tag{1}\\
& = \begin{cases}\boldsymbol{E}(\boldsymbol{r}) & \boldsymbol{r} \text { outside } V \\
\boldsymbol{0} & \boldsymbol{r} \text { inside } V\end{cases}
\end{align*}
$$

where $k$ is the wave number, $\eta_{0}$ is the free-space wave impedance, $\hat{\boldsymbol{n}}$ is the unit normal to $\partial V$ directed out of $V$ as shown in Figure 1 and the scalar Green function is

$$
g\left(k,\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|\right)=\frac{e^{i k\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|}}{4 \pi\left|\boldsymbol{r}-\boldsymbol{r}^{\prime}\right|}
$$

By asymptotic analysis where $|\boldsymbol{r}| \rightarrow \infty$, the above integral representation is simplified to the corresponding far field amplitude as (see e.g. [5])

$$
\left.\begin{array}{rl}
\boldsymbol{F}(\hat{\boldsymbol{r}})=i \frac{k^{2}}{4 \pi} & \hat{\boldsymbol{r}} \times \iint_{\partial V}
\end{array} \quad \hat{\boldsymbol{n}}\left(\boldsymbol{r}^{\prime}\right) \times \boldsymbol{E}\left(\boldsymbol{r}^{\prime}\right)\right] \text { ( } \quad \begin{aligned}
&  \tag{2}\\
&\left.-\eta_{0} \hat{\boldsymbol{r}} \times\left(\hat{\boldsymbol{n}}\left(\boldsymbol{r}^{\prime}\right) \times \boldsymbol{H}\left(\boldsymbol{r}^{\prime}\right)\right)\right] e^{-i k \hat{r}, \boldsymbol{r}^{\prime}} d S^{\prime}
\end{aligned}
$$

Ray-tube integration Exit aperture integration


Figure 2. Ray-tube versus exit aperture integration.
where the electrical field is given by

$$
\boldsymbol{E}(\boldsymbol{r})=\frac{e^{i k r}}{k r} \boldsymbol{F}(\hat{\boldsymbol{r}})
$$

The next step is to simplify the far field integral for a raytube. The electric and magnetic fields of the $n$ :th ray-tube are denoted $\boldsymbol{E}_{n}$ and $\boldsymbol{H}_{n}$, respectively. In GO, the electric and magnetic fields are related through the plane wave relation:

$$
\eta_{0} \boldsymbol{H}_{n}=\hat{\boldsymbol{k}}_{n} \times \boldsymbol{E}_{n}
$$

where $\hat{\boldsymbol{k}}_{n}$ is the direction (unit) vector of the $n$ :th center ray. For the case of ray-tube integration, $\hat{\boldsymbol{n}}=\hat{\boldsymbol{k}}_{n}$, and the far field integral is simplified to

$$
\begin{align*}
\boldsymbol{F}_{n}(\hat{\boldsymbol{r}}) & =i \frac{k^{2}}{4 \pi} \hat{\boldsymbol{r}} \times \iint_{S_{n}}\left[\hat{\boldsymbol{k}}_{n} \times \boldsymbol{E}_{n}+\hat{\boldsymbol{r}} \times \boldsymbol{E}_{n}\right] e^{-i k \hat{r} \cdot r^{\prime}} d S^{\prime}  \tag{3}\\
& \approx i \frac{A_{n} k^{2}}{4 \pi} \hat{\boldsymbol{r}} \times\left[\left(\hat{\boldsymbol{k}}_{n}+\hat{\boldsymbol{r}}\right) \times \boldsymbol{E}_{n}\right] e^{-i \boldsymbol{k} \hat{\boldsymbol{r}} \cdot r_{n}}
\end{align*}
$$

where $S_{n}$ is the cross-sectional surface of the ray-tube, which is orthogonal to the ray vector $\hat{\boldsymbol{k}}_{n}, A_{n}$ is the area of $S_{n}$ and $\boldsymbol{r}_{n}$ is the intersection point between the center ray and $S_{n}$ (the exit aperture). For the case of exit aperture integration, generally $\hat{\boldsymbol{n}} \neq \hat{\boldsymbol{k}}_{n}$ as shown in Figure 2 and the far field integral simplifies to

$$
\begin{align*}
\boldsymbol{F}_{n}(\hat{\boldsymbol{r}}) \approx i \frac{A_{n}^{\prime} k^{2}}{4 \pi} \hat{\boldsymbol{r}} & \times\left[\hat{\boldsymbol{n}} \times \boldsymbol{E}_{n}\right.  \tag{4}\\
& \left.-\hat{\boldsymbol{r}} \times\left(\hat{\boldsymbol{n}} \times\left(\hat{\boldsymbol{k}}_{n} \times \boldsymbol{E}_{n}\right)\right)\right] e^{-i k \hat{\boldsymbol{r}} \cdot \boldsymbol{r}_{n}}
\end{align*}
$$

where the projected surface area is

$$
A_{n}^{\prime}=\frac{A_{n}}{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}_{n}}
$$

The total far field amplitude is obtained by adding the individual contributions as

$$
\boldsymbol{F}^{\mathrm{SBR}}(\hat{\boldsymbol{r}})=\sum_{n} \boldsymbol{F}_{n}(\hat{\boldsymbol{r}})
$$

where $\boldsymbol{F}_{n}$ is computed with ray-tube integration according to (3) or exit aperture integration according to (4).

In the next subsection, the ray-tube and exit aperture integration formulas are compared.

### 2.2 Ripple anomaly of ray-tube integration

In this subsection, an electrical dipole source located inside a spherical surface is analysed with SBR, see Figure 3. The electric and magnetic fields for an electrical dipole located at the origin and oriented along the $z$-axis are [5]

$$
\left\{\begin{array}{rl}
\boldsymbol{E}_{0}(\boldsymbol{r})=\frac{p k^{2}}{\varepsilon_{0}} \frac{e^{i k r}}{4 \pi r}\{[3 \hat{\boldsymbol{r}}(\hat{z} \cdot \hat{\boldsymbol{r}})-\hat{z}]( & \left.\frac{1}{k^{2} r^{2}}-\frac{i}{k r}\right)  \tag{5}\\
& +\hat{\boldsymbol{r}} \times(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{r}})\}
\end{array}, \quad \begin{array}{l}
\boldsymbol{H}_{0}(\boldsymbol{r})=-i k \omega p \frac{e^{i k r}}{4 \pi r}\left(i-\frac{1}{k r}\right) \hat{\boldsymbol{r}} \times \hat{\boldsymbol{z}}
\end{array}\right.
$$

where $p$ is the dipole moment. The corresponding far fields are obtained by noticing that $(k r)^{-1}$ and $(k r)^{-2}$ vanish as $r \rightarrow \infty$, giving

$$
\left\{\begin{array}{l}
\boldsymbol{E}_{0}(\boldsymbol{r})=\frac{p k^{2}}{\varepsilon_{0}} \frac{e^{i k r}}{4 \pi r} \hat{\boldsymbol{r}} \times(\hat{z} \times \hat{\boldsymbol{r}})  \tag{6}\\
\boldsymbol{H}_{0}(\boldsymbol{r})=k \omega p \frac{e^{i k r}}{4 \pi r} \hat{\boldsymbol{r}} \times \hat{\boldsymbol{z}}
\end{array}\right.
$$

The dipole moment is selected to $p=4 \pi \varepsilon_{0} / k^{3}$ to normalize the far field amplitude. Notice that the above far fields fulfil the GO relation

$$
\eta_{0} \boldsymbol{H}_{0}(\boldsymbol{r})=\hat{\boldsymbol{r}} \times \boldsymbol{E}_{0}(\boldsymbol{r})
$$

To establish starting points for the rays, the dipole is enclosed in a spherical surface with radius $\rho$, which is sliced equidistantly in the spherical angles $\theta$ and $\phi$ through

$$
\begin{cases}\theta_{m}=m \Delta_{\theta}=m \pi /(M+1) & \text { where } 1 \leq m \leq M \\ \phi_{m n}=(n-1) \Delta_{\phi_{m}}=(n-1) 2 \pi / N_{m} & \text { where } 1 \leq n \leq N_{m}\end{cases}
$$

where $M$ and $N_{m}$ are selected to provide sufficiently small integration areas at the exit aperture. With the above spherical angles, the starting point of the corresponding center rays launched from the source is given by

$$
\boldsymbol{r}_{m n}=\rho \hat{\boldsymbol{r}}\left(\theta_{m}, \phi_{m n}\right)
$$

The cross-sectional area of the ray-tube is at the starting point $\boldsymbol{r}_{m n}$ given by

$$
\begin{aligned}
A_{m n}^{(1)} & =\int_{\theta_{m}-\Delta_{\theta} / 2}^{\theta_{m}+\Delta_{\theta} / 2} \int_{\phi_{m, n}-\Delta_{\phi_{m}}}^{\phi_{m, n}+\Delta_{\phi_{m}}} \rho^{2} \sin \theta d \theta d \phi \\
& =\rho^{2} \sin \theta_{m} 2 \sin \left(\Delta_{\theta} / 2\right) \Delta_{\phi_{m}}
\end{aligned}
$$

Note that $2 \sin \left(\Delta_{\theta} / 2\right) \approx \Delta_{\theta}$ for small $\Delta_{\theta}$. The total surface area is given by

$$
\sum_{m=1}^{M} \sum_{n=1}^{N_{m}} A_{m n}^{(1)} \approx 4 \pi \rho^{2}
$$

The above electrical dipole and its sphere is now translated from the origin to $\hat{z} a / 2$. Hence, the starting points of the translated source are given by $\boldsymbol{r}_{m n}^{(1)}=\boldsymbol{r}_{m n}+\hat{\boldsymbol{z}} a / 2$


Figure 3. The radius of the exit aperture sphere is $a$ and the dipole source illuminating the exit aperture is located at $\hat{z} a / 2$. The small sphere enclosing the dipole provides the starting points for the rays launched from the dipole source.
and the corresponding ray vectors are $\hat{\boldsymbol{k}}_{m n}=\hat{\boldsymbol{r}}_{m n}$, where $\hat{\boldsymbol{r}}_{m n}=\boldsymbol{r}_{m n} /\left|\boldsymbol{r}_{m n}\right|$, while the electric field at the starting point is given by (6) as

$$
\boldsymbol{E}_{m n}^{(1)}=\boldsymbol{E}_{0}\left(\boldsymbol{r}_{m n}\right)
$$

The electric field, according to GO, at a travel distance $s$ from the starting point is given by [1]

$$
\boldsymbol{E}_{m n}^{(2)}(s)=\boldsymbol{E}_{m n}^{(1)} \sqrt{\frac{\rho_{1} \rho_{2}}{\left(\rho_{1}+s\right)\left(\rho_{2}+s\right)}} e^{i k s}=\boldsymbol{E}_{0}\left(\boldsymbol{r}_{m n}\right) \frac{\rho}{\rho+s} e^{i k s}
$$

where $\rho_{1}=\rho_{2}=\rho$ are the radii of curvature of the wave front. Finally, the dipole source is enclosed in a spherical exit aperture centered at the origin and with radius $a$ as shown in Figure 3. The intersection point between the ray and the large sphere is found by solving for $s_{m}>0$ in the following equations

$$
\left\{\begin{array}{l}
\boldsymbol{r}=\boldsymbol{r}_{m n}+\hat{z} a / 2+\hat{\boldsymbol{r}}_{m n} s_{m} \\
|\boldsymbol{r}|=a
\end{array}\right.
$$

which gives

$$
s_{m}=\frac{a}{2}\left(\sqrt{3+\cos \theta_{m}}-\cos \theta_{m}\right)
$$

where the positive square root has been selected to yield a positive travel distance. The cross-sectional area of the ray-tube at the exit aperture, denoted $A_{m n}^{(2)}$, is determined by conserving the total power in the ray-tube, i.e.

$$
\left|\boldsymbol{E}_{m n}^{(1)}\right|^{2} \boldsymbol{A}_{m n}^{(1)}=\left|\boldsymbol{E}_{m n}^{(2)}\left(s_{m}\right)\right|^{2} A_{m n}^{(2)}
$$

Figure 4 shows the $\theta$-component of the far field amplitude computed with SBR according to (3) and exit aperture integration according to (4). The solid line is the exact result given by (6). The radius of the sphere is $a$ and $k a=20 \pi$, where $k$ is the wavelength.

### 2.3 Effect of holes in the integration surface

In the previous section, it was demonstrated that ray-tube integration can cause erroneous ripple in the far field, which


Figure 4. The $\theta$-component of the far field amplitude computed with ray-tube integration according to (3) and exit aperture integration according to (4). The radius of the sphere is $a=0.3 \mathrm{~m}$ and the dipole is displaced $a / 2$ from the center of the sphere, see Figure 3.
is not seen when using exit aperture integration. Considering the entire effective integration surface of Section 2.2, it is obvious that exit aperture integration yields a closed effective integration surface (i.e. the exit aperture sphere). However, using ray-tube integration, the effective integration surface contains holes in between different $\theta_{m}$-angles. Notice that no power is flowing through these holes, as the unit normal vectors of the holes are orthogonal to the Poynting vectors.

To explore the effect of (zero power flow) holes in the integration surface, a dipole enclosed in open and closed Huygens' surfaces is now considered. The dipole is located at the origin and is enclosed inside two hemispheres centered at the origin, with different radii, see Figure 5. The $\boldsymbol{E}$ and $\boldsymbol{H}$ from the enclosed dipole is evaluated according to (6) on the open and closed surface, respectively, and the far field is then calculated by (2). The result is shown in Figure 5, where clearly even small holes with zero power flow may cause erroneous ripple in the far field amplitude. Notice the small error in the integral representation result for the closed surface. The error is due to the fact that the integrated fields are far field approximations given by (6). The error vanishes if the radii of the hemispheres are sufficiently increased or if the total fields given by (5) are integrated.

To further explore the effect of holes in the Huygens' surfaces, the near fields are calculated by the integral representations (1) for the open and closed surface, respectively. In this case, the $\boldsymbol{E}$ and $\boldsymbol{H}$ are the total fields given by (5). The radii of the upper and lower sphere are $4.75 \lambda$ and $5.25 \lambda$, respectively, where $\lambda$ is the free-space wavelength. The magnitude of the Poynting vector is evaluated over the dashed scan line shown in Figure 6. It is seen in Figure 6 that the null field is distinct for the closed surface, but not for the open surface which is lacking a strict definition of when the point of observation is inside the volume $V$, where the extinction theorem of (1) produces the null field.


Figure 5. The far field amplitude for an open and closed surface, respectively. The dipole moment is chosen to normalize the far field amplitude. The small error in the integral representation result for the closed surface is due to the fact that the integrated $\boldsymbol{E}$ and $\boldsymbol{H}$ are far field approximations.


Figure 6. The magnitude of the Poynting vector evaluated by the integral representations (1). The dipole axis is directed out of the paper and the dipole moment is selected such that the power is one at the line of observation.

## 3 Conclusion

In this paper, it was demonstrated that ray-tube integration can cause erroneous ripple in the far field, which is not seen when using exit aperture integration. It was illustrated that the ripple is caused by omitting parts of the integration surface from which there is no power flow. It is concluded that exit aperture integration should be selected in SBR analysis.

Furthermore, it was demonstrated that the ripple is arising from the integral representations (1) when the integration surface is not closed to form the entire boundary of the considered volume $V$. Hence, Huygens' surfaces should be closed in general, as even holes from which there is no power flow may give ripple in the evaluated near and far fields.

## 4 Acknowledgements

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## References

[1] H. Ling, R. Chou and S. W. Lee, "Shooting and bouncing rays: Calculating RCS of an arbitrary cavity," IEEE Trans. on Antennas and Propagation, Vol. AP-37, No. 2, pp. 194-205, February 1989.
[2] R. Brem and T. F. Eibert, "Shooting and bouncing ray (SBR) modeling framework involving dielectrics and perfect conductors," IEEE Trans. on Antennas and Propagation, Vol. 63, No. 8, pp. 3599-3609, August 2015.
[3] S. Poulsen, "Shooting and Bouncing Rays in Radomes," 2019 International Conference on Electromagnetics in Advanced Applications (ICEAA), IEEE, pp. 449-452, 2019.
[4] S. W. Lee, H. Ling and R. Chou, "Ray-tube integration in shooting and bouncing ray method," Microwave and Optical Technology Letters, Vol. 1, No. 8, pp. 286-289, October 1988.
[5] Gerhard Kristensson, Scattering of Electromagnetic Waves by Obstacles, SciTech Publishing, 2016.

