# Numerical method for studying the spectrum of surface waves of an open inhomogeneous rectangular dielectric waveguide 

Maxim Snegur*(1) and Eugene Smolkin*(1)

(1) Penza State University, Penza, Russia.


#### Abstract

The problem of surface waves in a regular open waveguide of rectangular cross section is considered, which is reduced to a boundary value problem for the longitudinal components of the electromagnetic field in Sobolev spaces. To determine the solution, a variational formulation of the problem is used. A technique based on the combination of a version of the projection (Galerkin's) method and parameter shooting is proposed and applied for practical calculations.


## 1 Introduction

A significant class of vector problems of theoretical electromagnetics is the analysis of the electromagnetic wave propagation in open waveguides investigated by many authors [1,2,3]. Open dielectric waveguides [4] constitute here an important particular family. However, for open (unshielded) structures, and particularly for dielectric waveguides, a sufficiently complete theory of wave propagation has not been constructed so far. This paper presents the results of numerical study of the spectrum of propagating surface waves of an open rectangular waveguide with asymptotic conditions at infinity.

Note that in this paper we consider only the waves decaying with respect to the distance from the waveguide (imposing the corresponding conditions at infinity). Other types of waves are not taken into account.

## 2 Statement of the problem

Consider three-dimensional space $\mathbb{R}^{3}$ with the Cartesian coordinate system $O x y z$ filled with an isotropic medium without sources (vacuum) having the permittivity $\widetilde{\varepsilon}=\varepsilon_{0} \equiv$ const and permeability $\widetilde{\mu}=\mu_{0}$ equivconst. We will consider a mathematical model of the wave propagation in a regular (along the $O z$-axis) waveguide structure, the cross section of which by the plane $z=$ const forms a bounded region $\Omega$,

$$
\Omega:=\{\mathrm{x}=(x, y):(\mathrm{x}):|x|<a,|y|<a\}
$$

The nonhomogeneous dielectric waveguide under study is filled with a homogeneous isotropic material having the permittivity and permeability $\widetilde{\varepsilon}=\varepsilon(x, y), \operatorname{Im} \varepsilon=0$, and $\widetilde{\mu}=$
$\mu(\mathrm{x}), \operatorname{Im} \mu=0$, respectively. We assume that $\varepsilon(\mathrm{x}), \mu(\mathrm{x}) \in$ $C^{1}(\bar{\Omega})$ and $\min \varepsilon>\varepsilon_{0}, \min \mu>\mu_{0}$.

In the entire space, the permittivity and permeability are governed by the relations

$$
\tilde{\varepsilon}=\left\{\begin{array}{lr}
\varepsilon(\mathrm{x}), & \mathrm{x} \in \Omega,  \tag{1}\\
\varepsilon_{0}, & \mathrm{x} \in \mathbb{R}^{2} \backslash \bar{\Omega},
\end{array} \quad \tilde{\mu}=\left\{\begin{array}{lr}
\mu(\mathrm{x}), & \mathrm{x} \in \Omega \\
\mu_{0}, & \mathrm{x} \in \mathbb{R}^{2} \backslash \bar{\Omega}
\end{array}\right.\right.
$$

Determination of electromagnetic waves in a waveguide is the problem of finding nontrivial propagating wave solutions to the homogeneous system of Maxwell's equations, i.e., the solutions with dependence $e^{i \gamma z}$ on coordinate $z$ [4],

$$
\left\{\begin{array}{l}
\operatorname{rot} \mathbf{H}=-i \omega \widetilde{\varepsilon} \mathbf{E}  \tag{2}\\
\operatorname{rot} \mathbf{E}=i \omega \widetilde{\mu} \mathbf{H}
\end{array}\right.
$$

and

$$
\begin{align*}
\mathbf{E} & =\left(E_{x}(\mathrm{x}) \mathbf{e}_{\mathbf{x}}+E_{y}(\mathrm{x}) \mathbf{e}_{\mathbf{y}}+E_{z}(\mathrm{x}) \mathbf{e}_{\mathbf{z}}\right) e^{i \gamma z}  \tag{3}\\
\mathbf{H} & =\left(H_{x}(\mathrm{x}) \mathbf{e}_{\mathbf{x}}+H_{y}(\mathrm{x}) \mathbf{e}_{\mathbf{y}}+H_{z}(\mathrm{x}) \mathbf{e}_{\mathbf{z}}\right) e^{i \gamma z} \tag{4}
\end{align*}
$$

the transmission conditions for the tangential electric and magnetic field components on the surfaces of the "breaks" of permittivity and permeability,

$$
\begin{equation*}
\left.\left[\mathrm{E}_{\tau}\right]\right|_{\bar{\Omega}}=0,\left.\left[\mathrm{H}_{\tau}\right]\right|_{\bar{\Omega}}=0 \tag{5}
\end{equation*}
$$

and the radiation condition at infinity: the electromagnetic field $\mathbf{E}, \mathbf{H} \rightarrow O\left(\frac{1}{\sqrt{r}}\right), r \rightarrow \infty, r=\sqrt{x^{2}+y^{2}}$.

It is known [5, 6, 7] that the field in a waveguide can be represented using two scalar functions:

$$
\Pi:=E_{z}(\mathrm{x}), \Phi:=H_{z}(\mathrm{x})
$$

The problem of the determination of electromagnetic surface waves propagating in the inhomogeneous waveguide of rectangular cross-section consists in finding such $\gamma \in \mathbb{R}$ for which there exist nontrivial solutions of the following system of differential equations

$$
\left\{\begin{array}{l}
\Delta \Pi-\widetilde{\kappa}^{2} \Pi=-\left(\frac{\nabla \widetilde{\varepsilon}}{\widetilde{\varepsilon}}+\frac{\omega^{2}}{\widetilde{\kappa}^{2}} \nabla \widetilde{\varepsilon} \widetilde{\mu}\right) \nabla \Pi-\frac{\gamma \omega}{\widetilde{\varepsilon}^{\widetilde{\kappa}^{2}}} J(\widetilde{\varepsilon} \widetilde{\mu}, \Phi) \\
\Delta \Phi-\widetilde{\kappa}^{2} \Phi=-\left(\frac{\nabla \widetilde{\mu}}{\widetilde{\mu}}+\frac{\omega^{2}}{\widetilde{\kappa}^{2}} \nabla \widetilde{\varepsilon} \widetilde{\mu}\right) \nabla \Phi+\frac{\gamma \omega}{\widetilde{\mu} \widetilde{\kappa}^{2}} J(\widetilde{\varepsilon} \widetilde{\mu}, \Pi)
\end{array}\right.
$$

where $\widetilde{\kappa}_{0}^{2}=\gamma^{2}-\omega^{2} \widetilde{\varepsilon} \widetilde{\mu}$ and

$$
J(u, v):=\frac{\partial u}{\partial x} \frac{\partial v}{\partial y}-\frac{\partial u}{\partial y} \frac{\partial v}{\partial x},
$$

satisfying the boundary conditions

$$
\begin{gathered}
{\left.[\Pi]\right|_{\bar{\Omega}}=0,\left.[\Phi]\right|_{\bar{\Omega}}=0} \\
\left.\gamma\left[\frac{1}{\widetilde{\kappa}^{2}} \frac{\partial \Phi}{\partial \tau}\right]\right|_{\bar{\Omega}}+\left.\left[\frac{\omega \widetilde{\varepsilon}}{\widetilde{\kappa}^{2}} \frac{\partial \Pi}{\partial n}\right]\right|_{\bar{\Omega}}=0, \\
\left.\gamma\left[\frac{1}{\widetilde{\kappa}^{2}} \frac{\partial \Pi}{\partial \tau}\right]\right|_{\bar{\Omega}}-\left.\left[\frac{\omega \tilde{\mu}}{\widetilde{\kappa}^{2}} \frac{\partial \Phi}{\partial n}\right]\right|_{\bar{\Omega}}=0,
\end{gathered}
$$

the energy boundedness condition in $\Omega$

$$
\int_{\Omega}\left(|\nabla \Pi|^{2}+|\nabla \Phi|^{2}+|\Pi|^{2}+|\Phi|^{2}\right) d \mathrm{x}<\infty
$$

and the asymptotic radiation condition at infinity:

$$
\begin{equation*}
\left.\frac{\partial \Pi}{\partial n}\right|_{\bar{\Omega}_{\infty}}+\left.\kappa_{0} \Pi\right|_{\bar{\Omega}_{\infty}}=0,\left.\quad \frac{\partial \Phi}{\partial n}\right|_{\bar{\Omega}_{\infty}}+\left.\kappa_{0} \Phi\right|_{\bar{\Omega}_{\infty}}=0 \tag{6}
\end{equation*}
$$

where $\quad \kappa_{0}^{2}=\gamma^{2}-\omega^{2} \varepsilon_{0} \mu_{0} \quad$ and $\quad \bar{\Omega}_{\infty}:=$ $\{(x, y):|x|<b,|y|<b\}$ is the boundary of a large enough domain $\Omega_{\infty}$ such that condition (6) is satisfied.
Remark 1. The solution of the latter system in free space is determined by the following series

$$
\left\{\begin{array}{l}
\Pi(r)=\sum_{m=1}^{\infty} C_{m} \cos m \phi K_{m}\left(\kappa_{0} r\right), \mathrm{x} \in \mathbb{R}^{2} \backslash \bar{\Omega}, \\
\Phi(r)=\sum_{m=1}^{\infty} \widetilde{C}_{m} \cos m \phi K_{m}\left(\kappa_{0} r\right), \mathrm{x} \in \mathbb{R}^{2} \backslash \bar{\Omega}
\end{array}\right.
$$

where $K_{m}$ is a modified Bessel function (Mcdonald function), $r=\sqrt{x^{2}+y^{2}}, \phi=\arctan \frac{y}{x}, C_{m}$ and $\widetilde{C}_{m}$ are constant. For sufficiently large values of the argument $r$, the following asymptotics is valid [8]

$$
\frac{u(r)}{u_{r}^{\prime}}=-\frac{1}{\kappa_{0}}+O\left(\frac{1}{r}\right) .
$$

The resulting asymptotics allows one to introduce asymptotic radiation conditions.

## 3 Numerical Method and Results

The weak formulation [7] of the problem under consideration leads to the following variational relation:

$$
\begin{gather*}
\int_{\Omega_{\infty}}(\widetilde{\varepsilon} \Pi \bar{u}+\widetilde{\mu} \Phi \bar{v}) d \mathrm{x}+\int_{\Omega_{\infty}} \frac{\widetilde{\varepsilon} \nabla \Pi \nabla \bar{u}+\widetilde{\mu} \nabla \Phi \nabla \bar{v}}{\widetilde{\kappa}^{2}} d \mathrm{x}+ \\
\quad+\left.\int_{\bar{\Omega}_{\infty}} \frac{\varepsilon_{0} \Pi \bar{u}+\mu_{0} \Phi \bar{v}}{\kappa_{0}}\right|_{\bar{\Omega}_{\infty}} d \tau+ \\
\quad+\int_{\Omega} \frac{\gamma \omega}{\widetilde{\kappa}^{4}}(\bar{v} J(\widetilde{\mu}, \Pi)-\widetilde{\varepsilon} \bar{u} J(\widetilde{\varepsilon} \widetilde{\mu}, \Phi)) d \mathrm{x}- \\
-\left.\int_{\bar{\Omega}} \gamma \omega \frac{\varepsilon \mu-\varepsilon_{0} \mu_{0}}{\kappa_{0}^{2} \kappa^{2}}\left(\frac{\partial \Pi}{\partial \tau} \bar{v}-\frac{\partial \Phi}{\partial \tau} \bar{u}\right)\right|_{\bar{\Omega}} d \tau=0 . \tag{7}
\end{gather*}
$$

Using the projection method we reduce the addressed variational relation to a system of algebraic equations.

We split $\Omega_{\infty}$ into $n$ identical squares with sidelength $h_{0}$ and take bilinear functions with rectangular supports as basis functions $\psi_{i}$ on $\Omega_{\infty}$. The support $\Psi_{i}$ of the basis function $\psi_{i}$ is the union of four squares with a common point $\mathrm{x}_{i}^{0}$. In addition, we assume that the centers $\mathrm{x}_{i}^{0}$ of the base elements $\Psi_{i}$ lie inside the region $\bar{\Omega}_{\infty}$, so that $\mathrm{x}_{i}^{0} \in \bar{\Omega}_{\infty}$,

$$
\begin{equation*}
\psi_{i}(\mathrm{x})=\psi_{i}^{x}(x) \psi_{i}^{y}(y) \tag{8}
\end{equation*}
$$

and

$$
\begin{gather*}
\psi_{i}^{x}(x)=\left\{\begin{array}{l}
\frac{x-x_{i}^{0}+h}{h}, x \in\left(x_{i}^{0}-h, x_{i}^{0}\right), \\
\frac{x_{i}^{0}+h-x}{h}, x \in\left(x_{i}^{0}, x_{i}^{0}+h\right), \\
0, x \notin\left(x_{i}^{0}-h, x_{i}^{0}+h\right),
\end{array}\right.  \tag{9}\\
\psi_{i}^{y}(y)=\left\{\begin{array}{c}
\frac{y-y_{i}^{0}+h}{h}, y \in\left(y_{i}^{0}-h, y_{i}^{0}\right), \\
\frac{y_{i}^{0}+h-y}{h}, y \in\left(y_{i}^{0}, y_{i}^{0}+h\right), \\
0, y \notin\left(y_{i}^{0}-h, y_{i}^{0}+h\right),
\end{array}\right. \tag{10}
\end{gather*}
$$

Such basis functions take into account the physical nature of the problem under consideration.

We look for an approximate solution as a finte sum with real coefficients $\alpha_{i}$ and $\beta_{j}$ such that

$$
\begin{equation*}
\Pi=\sum_{i=1}^{n+1} \alpha_{i} \psi_{i}, \quad \Phi=\sum_{j=1}^{n+1} \beta_{j} \psi_{j} . \tag{11}
\end{equation*}
$$

Substituting functions $\Pi$ and $\Phi$ with representations (11) into the variational relation, we obtain a system of linear equations with respect to $\alpha_{i}$ and $\beta_{j}$ (for a fixed value of $\gamma$ )

$$
\begin{equation*}
A(\gamma) x=0 \tag{12}
\end{equation*}
$$

where matrices $A(\gamma)$ and $x$ have the form

$$
A=\left(\begin{array}{cccccc}
A_{e e}^{1,1} & \cdots & A_{e e}^{1, n+1} & A_{e m}^{1,1} & \cdots & A_{e m}^{1, n+1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_{e e}^{n+1,1} & \cdots & A_{e e}^{n+1, n+1} & A_{e m}^{n+1,1} & \cdots & A_{e m}^{n+1, n+1} \\
A_{m e}^{1,1} & \cdots & A_{m e}^{1, n+1} & A_{m m}^{1,1} & \cdots & A_{m m}^{1, n+1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_{m e}^{n+1,1} & \cdots & A_{m e}^{n+1, n+1} & A_{m m}^{n+1,1} & \cdots & A_{m m}^{n+1, n+1}
\end{array}\right),
$$

and

$$
x=\left(\alpha_{1} \cdots \alpha_{n+1} \beta_{1} \cdots \beta_{n+1}\right)^{T}
$$

where

$$
\begin{aligned}
A_{e e}^{i, j}=\int_{\Psi_{i}} \widetilde{\varepsilon} \psi_{i} \psi_{j} d \mathrm{x} & +\int_{\Psi_{i}} \frac{\widetilde{\varepsilon} \nabla \psi_{i} \nabla \psi_{j}}{\widetilde{\kappa}^{2}} d \mathrm{x}+ \\
& +\left.\int_{\bar{\Omega}_{\infty}} \frac{\varepsilon_{0}}{\kappa_{0}} \psi_{i} \psi_{j}\right|_{\overline{\Omega_{\infty}}} d \tau, i, j=\overline{1, n+1},
\end{aligned}
$$

$$
\begin{aligned}
& A_{e m}^{i, j}= \int_{\Psi_{i}} \frac{\gamma \omega}{\widetilde{\kappa}^{4}} \psi_{j} J\left(\widetilde{\varepsilon} \widetilde{\mu}, \psi_{i}\right) d \mathrm{x}- \\
&-\left.\int_{\bar{\Omega}_{\infty}} \gamma \omega \frac{\varepsilon \mu-\varepsilon_{0} \mu_{0}}{\kappa^{2} \kappa_{0}^{2}} \frac{\partial \psi_{i}}{\partial \tau} \psi_{j}\right|_{\bar{\Omega}} d \tau, i=\overline{1, n+1}, j=\overline{1, n+1} \\
& A_{m e}^{i, j}=-\int_{\Psi_{i}} \frac{\gamma \omega}{\widetilde{\kappa}^{4}} \psi_{j} J\left(\widetilde{\varepsilon} \widetilde{\mu}, \psi_{i}\right) d \mathrm{x}+ \\
&+\left.\int_{\Omega_{\infty}} \gamma \omega \frac{\varepsilon \mu-\varepsilon_{0} \mu_{0}}{\kappa^{2} \kappa_{0}^{2}} \frac{\partial \psi_{i}}{\partial \tau} \psi_{j}\right|_{\bar{\Omega}} d \tau, i=\overline{1, n+1}, j=\overline{1, n+1} \\
& A_{m m}^{i, j}=\int_{\Psi_{i}} \widetilde{\varepsilon} \psi_{i} \psi_{j} d \mathrm{x}+\int_{\Psi_{i}} \frac{\widetilde{\varepsilon} \nabla \psi_{i} \nabla \psi_{j}}{\widetilde{\kappa}^{2}} d \mathrm{x}+ \\
& \quad+\left.\int_{\frac{\Omega_{\infty}}{}} \frac{\varepsilon_{0}}{\kappa_{0}} \psi_{i} \psi_{j}\right|_{\overline{\Omega_{\infty}}} d \tau, i, j=\overline{1, n+1}
\end{aligned}
$$

Thus $A(\gamma)$ is a $2(n+1) \times 2(n+1)$ matrix.
Denote by $\Delta(\gamma)$ the determinant of $A(\gamma)$,

$$
\begin{equation*}
\Delta(\gamma)=\operatorname{det} A(\gamma) \tag{13}
\end{equation*}
$$

Definition of approximate solution. If there exists $\gamma=\widetilde{\gamma}$ such that $\Delta(\widetilde{\gamma})=0$, then $\widetilde{\gamma}$ is an approximate eigenvalue of the problem. In other words, if an interval $[\gamma, \gamma]$ is such that $\Delta(\gamma) \times \Delta(\gamma)<0$, then this means that there exists $\gamma=\widetilde{\gamma} \in$ $[\gamma, \gamma]$ which is a propagation constant of problem (2)-(5). $\overline{T h i s}$ value can be calculated with any prescribed accuracy.

As a model problem, consider the following set of parameters: $a=1, \varepsilon=9, \mu=1, \varepsilon_{0}=\mu_{0}=1$. Dispersion curves (graphs of the dependence of normalized propagation constant $\gamma / \omega$ on frequency $\omega$ ) are shown in the figure.


Figure 1. Dispersion curves.
Having fixed the frequency $\omega=0.5$, let us study the change in the value of the first normalized propagation constant $\gamma / \omega$ (indicated by the red dot in Fig. 1) from the number of base elements (NBE). The results are shown in the following table.

We see that with an increase in NBE, the accuracy of the eigenvalue calculation is enhanced (the result becomes more precise).

| NBE | 441 | 961 | 1681 | 3721 | 6561 | 10201 |
| :---: | :--- | :--- | :--- | :--- | :--- | :--- |
| $\gamma / \omega$ | 2.755 | 2.738 | 2.751 | 2.749 | 2.748 | 2.748 |
| $\gamma$ | 1.378 | 1.369 | 1.376 | 1.375 | 1.374 | 1.374 |

Table 1. Normalized eigenvalues of the problem with a change in the NBE.

## 4 Conclusion

The developed numerical method is efficient for the analysis of the wave propagation in open dielectric waveguides filled with inhomogeneous media and can be applied to calculating propagation constants of polarized waves in cylindrical circular waveguides and waveguides having noncoordinate cross-sections.

## 5 Acknowledgements

This work was supported by the RFBR, project number 20-31-70010.

## References

[1] V.V. Lozhechko, Yu.V. Shestopalov, "Problems of the excitation of open cylindrical resonators with an irregular boundary," Computational Mathematics and Mathematical Physics, 35 (1),1995, pp. 53-61.
[2] E.V. Chernokozhin, Yu.V. Shestopalov, "Mathematical methods for the study of wave scattering by open cylindrical structures," Radiotekhnika i Elektronika, 42 (11), 1997, pp. 1299-1311.
[3] A.S. Ilyinsky, G.Ya. Slepyan, A.Ya. Slepyan, "Propagation, Scattering and Dissipation of Electromagnetic Waves," Peter Peregrinus Ltd, 1993.
[4] A. W. Snyder, J. Love, "Optical waveguide theory," Springer,1983.
[5] Yu.G. Smirnov, Eu. Smolkin, "Discreteness of the spectrum in the problem on normal waves in an open inhomogeneous waveguide," Differential Equations, 53, 2017, pp. 1262-1273.
[6] Yu.G. Smirnov, E. Smolkin, M.O. Snegur, "Analysis of the Spectrum of Azimuthally Symmetric Waves of an Open Inhomogeneous Anisotropic Waveguide with Longitudinal Magnetization," Computational Mathematics and Mathematical Physics, 58(11), 2018, pp. 1887-1901.
[7] Yu.G. Smirnov, E. Smolkin, "Operator Function Method in the Problem of Normal Waves in an Inhomogeneous Waveguide," Differential Equations, 54(9), 2018, pp. 1262-1273.
[8] M. Abramowitz, I.A. Stegun, "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables" Applied Mathematics Series 55, 1983.

