URSI GASS 2021, Rome, Italy, 28 August - 4 September 2021

Nonlinear TE-polarized electromagnetic waves in a cylindrical waveguide

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Abstract

In this work, we study TE-polarized electromagnetic waves propagating in an inhomogeneous dielectric waveguide of circular cross section filled with a medium; the nonlinearity is simulated by Kerr's law. For the numerical solution of the problem, an iterative algorithm is proposed, and its convergence is proved. The existence of the roots of the dispersion equation, the propagation constants of the waveguide, is proved. The conditions are obtained when several waves can propagate, the localization regions of the corresponding propagation constants are determined.

1 Introduction

The propagation of a light beam in a homogeneous nonlinear medium or a waveguide structure filled with a nonlinear medium described by Kerr's law has been actively studied over the past two decades [1]–[8]. The propagation of TE-polarized waves in a three-layer lossless medium, one of the layers of which is filled with a nonlinear medium, was studied in detail [2, 3, 4, 6]. In these works, analytical solutions of the corresponding differential equations were obtained using elliptic functions and numerical results were presented. However, when studying other structures, for example, a circular dielectric waveguide, it is no longer possible to obtain analytical solutions; however, this goal may be achieved using numerical methods.

The results related to the propagation of TE-polarized waves in a nonlinear dielectric circular waveguide are presented in several works of H.-W. Shürmann, V.S.Serov and Yu. V. Shestopalov [5]. For the first time, the equations describing wave propagation in a nonlinear medium with nonlinearity expressed by Kerr's law were obtained in 1971–1972 in the pioneering works by P.N. Eleonskiy and V.P. Silin (see, for example, [9]).

An introduction to the theory of nonlinear guided waves in media with Kerr and Kerr-like nonlinearities is given in [10]. For purely nonlinear TE and TM waves one may refer to [11]-[16].

In this work, we study electromagnetic waves propagating in an inhomogeneous dielectric waveguide of circular cross section filled with a medium where nonlinearity is simulated by Kerr's law.

2 Statement of the problem

Let $\mathbb{R}^3$ denote the three-dimensional space equipped with cylindrical coordinate system $O\rho\varphi z$ and filled with an isotropic source-free medium (vacuum) having permittivity ε_0 and permeability μ_0. Consider the problem of determining nonlinear TE-polarized surface waves propagating in a cylindrical circular waveguide. The cross-section of the waveguide will be either a circle of radius $r > 0$, or a ring with an inner radius $r_0 > 0$ (of a perfectly conducting cylinder) and external radius $r > r_0$ (see Fig. 1). In this work we consider both waveguides setting in the first case $r_0 = 0$.

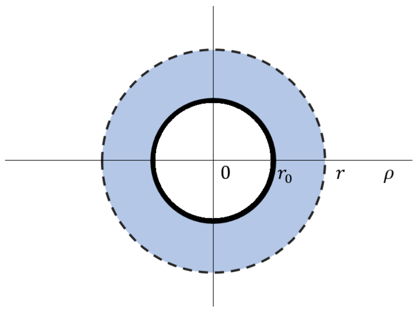

Figure 1. Geometry of the problem.

We assume that permittivity in the entire space has the form $\widetilde{\varepsilon}\varepsilon_0$, where

$$\widetilde{\varepsilon} = \begin{cases} \varepsilon + \alpha|\mathbf{E}|^2, & r_0 \leqslant \rho \leqslant r, \\ 1, & \rho > r, \end{cases} \tag{1}$$

$\alpha(\rho)$ and $\varepsilon(\rho)$ are continuous functions for $\rho \in [r_0, r]$ and $|\mathbf{E}|^2 = \left|(\mathbf{E}e^{-i\omega t}, \mathbf{e}_\rho)\right|^2 + \left|(\mathbf{E}e^{-i\omega t}, \mathbf{e}_\varphi)\right|^2 + \left|(\mathbf{E}e^{-i\omega t}, \mathbf{e}_z)\right|^2$; ω is the circular frequency.

We will consider TE-polarized waves in the harmonic mode (see [9]),

$$\mathbf{E}e^{-i\omega t} = e^{-i\omega t}(0, E_\varphi, 0)^T, \; \mathbf{H}e^{-i\omega t} = e^{-i\omega t}(H_\rho, 0, H_z)^T,$$

where $\mathbf{E}, \mathbf{H}$ are complex amplitudes and

$$E_\varphi = \mathrm{E}_\varphi(\rho)e^{i\gamma z}, \;\; H_\rho = \mathrm{H}_\rho(\rho)e^{i\gamma z}, \;\; H_z = \mathrm{H}_z(\rho)e^{i\gamma z}, \tag{2}$$

where γ is the spectral parameter.

The problem of finding TE-polarized waves in a circular waveguide can be reduced [11, 13, 15] to the following boundary value problem: *find quantities γ such that,* *for a given constant $\widetilde{C} \neq 0$, there is a nonzero function $u(\rho;\gamma) := \mathrm{E}_\varphi$ satisfying the boundary conditions*

$$u(r_0) = 0, \tag{3}$$

defined by the formula

$$u = \widetilde{C}K_1(\kappa\rho),\ \kappa = \sqrt{\gamma^2 - \kappa_0^2},\ \kappa_0^2 = \omega^2\varepsilon_0\mu_0,\ \rho > r, \tag{4}$$

and solving the equation

$$\left(\rho u'\right)' + \left(\kappa_0^2\rho\left(\varepsilon + \alpha u^2\right) - \frac{1}{\rho}\right)u - \gamma^2\rho u = 0, \tag{5}$$

for $r_0 \leq \rho \leq r$; moreover, function $u(\rho;\gamma)$ thus defined satisfies transmission conditions for surface waves: i.e., the electromagnetic field decays exponentially as $\rho \to \infty$ in the region $\rho > r$.

3 Numerical method and results

Denote $\lambda = \gamma^2$ and consider the following boundary value problem

$$(\rho u')' + \left(\widetilde{\varepsilon}\rho - \frac{1}{\rho}\right)u + \widetilde{\alpha}\rho u^3 = \lambda\rho u, \tag{6}$$

where $\widetilde{\varepsilon} = \kappa_0^2\varepsilon$ and $\widetilde{\alpha} = \kappa_0^2\alpha$, with the boundary conditions

$$\begin{aligned} u(r_0) = 0 \quad u'(r_0) = A(>0), r_0 \geq 0 \\ u'(r) = \beta u(r), \end{aligned} \tag{7}$$

where

$$\beta := \kappa\frac{K_1'(\kappa r)}{K_1(\kappa r)}, \quad \kappa^2 = \lambda - \kappa_0^2 \quad \beta < 0 \text{ for } \lambda - \kappa_0^2 > 0.$$

Next introduce a new function v such that

$$u = \frac{v}{\sqrt{\rho}}$$

and calculate the derivative to get

$$u' = \frac{v'}{\sqrt{\rho}} - \frac{v}{2\sqrt{\rho^3}}.$$

Multiplying the last equality by ρ we obtain

$$\rho u' = \sqrt{\rho}v' - \frac{v}{2\sqrt{\rho}}.$$

Differentiate the resulting expression

$$(\rho u')' = \sqrt{\rho}v'' + \frac{1}{4\sqrt{\rho^3}}v,$$

substitute to equation (6), and divide both sides by $\sqrt{\rho}$; as a result, we obtain

$$v'' + \left(\widetilde{\varepsilon}\frac{1}{\rho} - \frac{3}{4\rho^2}\right)v + \widetilde{\alpha}\frac{1}{\rho}v^3 = \lambda v.$$

Next, introducing the notations

$$q_0 = \left(\widetilde{\varepsilon}\frac{1}{\rho} - \frac{3}{4\rho^2}\right),\ \alpha_0 = \widetilde{\alpha}\frac{1}{\rho},$$

we obtain the following boundary value problem

$$v'' + q_0 v + \alpha_0 v^3 = \lambda v \tag{8}$$

with the bounary conditions

$$\begin{aligned} v'(r) = \beta_0 v(r) \\ v(r_0) = 0, \quad v'(r_0) = A\sqrt{r_0},\ \text{for } r_0 > 0, \end{aligned} \tag{9}$$

where

$$\beta_0 = \frac{1}{2r} + \beta \sim -\sqrt{\lambda} \text{ for } \lambda \to \infty. \tag{10}$$

The first integral of differential equation (8) has the form

$$v'^2 + (q_0 - \lambda)v^2 + \frac{\alpha_0}{2}v^4 = C, \quad C = A^2 > 0 \tag{11}$$

Statement 1. *If $u(r) = 0$, then the problem for TE waves in a cylindrical waveguide has an infinite spectrum.*

Statement 2. *For $r_0 > 0$, the problem for TE waves in a cylindrical waveguide has an infinite spectrum.*

Consider the right boundary condition in (9); taking a square, we get

$$\beta_0^2\lambda = \frac{1}{4r^2} + \frac{\beta}{r} + (\beta^2 - \lambda) = O(1) \text{ for } \lambda \to \infty.$$

Using the properties of the Macdonald function [17], we obtain

$$\begin{aligned}
\beta_0^2\lambda &= (\lambda - \kappa_0^2)\frac{K'^2_1(\kappa r)}{K_1^2(\kappa r)} - \lambda = \\
&= \lambda\left(\frac{K'^2_1(\kappa r)}{K_1^2(\kappa r)} - 1\right) - \kappa_0^2\frac{K'^2_1(\kappa r)}{K_1^2(\kappa r)} \sim \frac{\lambda}{\kappa r} - \kappa_0^2 = \\
&= \left\{\frac{K'_1(z)}{K_1(z)} + 1 = -\frac{1 + \dfrac{\mu+3}{8z}}{1 + \dfrac{\mu-1}{8z}} + 1 = \frac{-1 - \dfrac{\mu+3}{8z} + 1 + \dfrac{\mu-1}{8z}}{1 + \dfrac{\mu-1}{8z}} = \right. \\
&= \left.\frac{-4}{8z + \mu - 1} = \frac{-4}{8z+3} \sim -\frac{1}{2z}\right\} = \frac{\sqrt{\lambda}}{r} + O(1);
\end{aligned}$$

$$\frac{\lambda}{\kappa r} = \frac{\lambda - \kappa_0^2 + \kappa_0^2}{\kappa r} = \sqrt{\lambda - \kappa_0^2}\frac{1}{r} + \frac{\kappa_0^2}{\kappa r} = \sqrt{\lambda}\frac{1}{r} + O(1).$$

Now consider the biquadratic equation

$$\beta_0^2 v^2(r) + (q_0 - \lambda)v^2 + \frac{\alpha_0}{2}v^4 - C = 0.$$

Make the subsitution $v^2(r) = t \geq 0, \alpha_0 > 0$ to get the quadratic equation

$$\frac{\alpha_0}{2}t^2 + (\beta_0^2 + q_0 - \lambda)t - C = 0$$

with the discriminant $D=(\beta_0^2+q_0-\lambda)^2+2\alpha_0 C \quad (>0)$.

The roots of the quadratic equation are

$$t_{1,2}=\frac{\lambda-\beta_0^2-q_0\pm\sqrt{D}}{\alpha_0}$$

and

$$t_1=\frac{\lambda-\beta_0^2-q_0+\sqrt{D}}{\alpha_0}>0,\ t_2=\frac{\lambda-\beta_0^2-q_0-\sqrt{D}}{\alpha_0}<0.$$

Set

$$v(r)=\sqrt{\frac{\lambda-\beta_0^2-q_0+\sqrt{D}}{\alpha_0}}\quad \Big(=0(1),\ \lambda\to\infty\Big).$$

Thus, we obtain the *dispersion equation*

$$\int\limits_0^{v(r)} w dv+NT=r \tag{12}$$

where

$$T=\int\limits_{-v_1}^{v_1} w dv,$$

and the integrand i

$$w=\frac{1}{\sqrt{A^2 r_0+(\lambda-q_0)v^2-\dfrac{\alpha_0}{2}v^4}}.$$

A detailed study of this dispersion equation was performed in [18] where it was proved that it has infinitely many solutions.

An algorithm for finding approximate solutions of a nonlinear eigenvalue problem will be described below. To this end, we obtain the equation

$$F(\lambda):=NT-r=0,$$

where

$$T=2\int\limits_0^{v_1}\frac{dv}{\sqrt{A_0+(\lambda-q_0)v^2-\dfrac{\alpha_0}{2}v^4}},$$

and

$$v_1=\frac{\lambda-q_0+\sqrt{(q_0-\lambda)^2+2\alpha_0 A_0}}{\alpha_0}.$$

The computational algorithm of the numerical method is as follows:

1. Set the values $A_0, q_0, \kappa_0, \alpha_0, \lambda\in[\lambda_1;\lambda_2], N$;
2. Define number M of the partitions of segment $[\lambda_1;\lambda_2]$. Make a loop over $\lambda_i=\lambda_1+Hi$, where $H=(\lambda_2-\lambda_1)/M, i=0,\dots,M$;
3. With the known (inside the loop) λ_i, we calculate:
$$v_1=\frac{\lambda_i-q_0+\sqrt{(q_0-\lambda_i)^2+2\alpha_0 A_0}}{\alpha_0},$$
$$T=2\int\limits_0^{v_1}\frac{dv}{\sqrt{A_0+(\lambda_i-q_0)v^2-\dfrac{\alpha_0}{2}v^4}}$$
and
$$F(\lambda_i)=NT(\lambda_i)-r;$$
4. Check the condition $F(\lambda_i)F(\lambda_i-1)<0$, then $\lambda_i^*=\dfrac{\lambda_{i-1}+\lambda_i}{2}$ is a root;
5. If the condition is not satisfied, then we run the cycle further.

To exemplify application of the above computational algorithm, choose the following specific variant of the values of constants $A_0=1, q_0=1, \kappa_0=1,\ \alpha_0=0.01, \lambda_1=1, \lambda_2=3, N=1,2,3; M=40$ and find a solution to the problem.

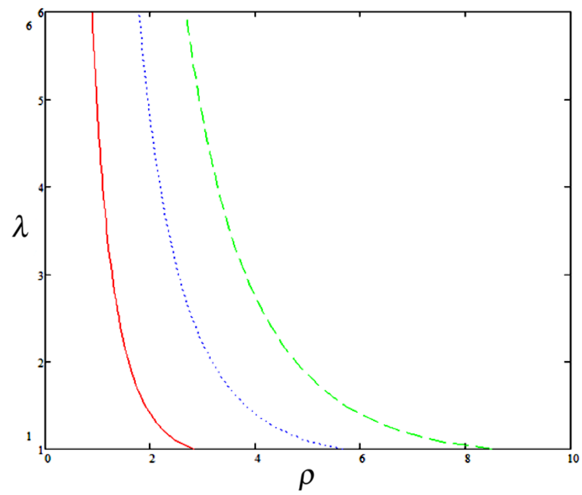

Figure 2. Numerical results. The graph of function $\lambda(\rho)$, where $N=1$ (red), $N=2$ (blue) and $N=3$ (green).

4 Conclusion

The developed numerical method can be efficiently applied for calculating TE-polarized electromagnetic waves in a cylindrical waveguide filled with Kerr-nonlinear medium and can be developed to determine numerically the propagation constants of polarized waves in cylindrical circular waveguides with more complicated nonlinearities.

5 Acknowledgements

This work was supported by the Russian Science Foundation, project no. 20-11-20087.

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