URSI GASS 2021, Rome, Italy, 28 August - 4 September 2021

A Simplified Derivation of Causality from Passivity for the Impedance Representation of Antennas

Arthur D. Yaghjian
Electromagnetics Research, Concord, MA 01742, Email: a.yaghjian@comcast.net

Abstract

A relatively simple, straightforward, self-contained derivation, sufficiently general for most engineering purposes, is given to show that passivity implies causality for the input-impedance representation of linear, single-port, time-invariant transmitting antennas.

1 Introduction

A powerful theorem for linear, time-invariant networks states that passivity of the network implies causality. This important theorem was first proven for input-impedance or input-admittance representations of networks by Youla et al. [1] using rigorous linear operator theory with Lebesgue measure and later by Zemanian [2, sec. 10.3] using Schwartz's approach to rigorous distribution theory. Both methods of proof are rather involved and removed from the physics, and require a substantial amount of preliminary mathematical development and analysis that may be prohibitive for the uninitiated, who nevertheless may be familiar with the definition of delta functions as a parametric limit of well-defined Riemann integrable functions.

Therefore, it is the main purpose of this communication to provide within the input-impedance representation of transmitting antennas a self-contained, sufficiently general derivation, using only the basic mathematical tools familiar to most of the antenna engineering community, to prove that passivity of a linear, time-invariant, single-port transmitting antenna implies that the antenna is also causal. We assume realistic time-domain voltages and currents that are effectively time-limited and whose corresponding frequency-domain voltages and currents are effectively bandlimited. In addition, all voltages and currents are assumed to be bounded and Riemann integrable in both the time and frequency domains. Linearity is defined by assuming the frequency-domain voltage is equal to the frequency-domain current multiplied by a frequency-domain input impedance. Within these assumptions of time-limited, bandlimited, bounded, Riemann integrable functions, all the important steps of the derivation are rigorously justified by referencing the relevant classical textbook theorems of differentiation and Riemannian integration. Although the basic definitions and analysis needed to prove causality from passivity are contained herein, the paper by Triverio et al. can be consulted for a review of the concepts of passivity, causality, and stability [3].

2 Voltage and Current of a Single-Port Transmitting Antenna

Consider the time-domain voltage $v(t)$ and current $i(t)$ for a single-port transmitting antenna shown in Fig. 1. A generator applies a voltage $v(t)$ or a current $i(t)$ to the input terminal of the port, resulting in a current $i(t)$ or a voltage $v(t)$, respectively, at the same terminal. Assume that these are realistic operational voltages and currents that are time-limited (they are zero or less than the noise level outside a finite interval), bounded, and Riemann integrable (and thus absolutely integrable [4, p. 116]). Consequently, their Fourier transforms exist to give the corresponding frequency-domain voltage $V(\omega)$ and current $I(\omega)$

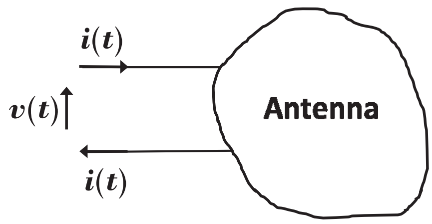

Figure 1. Schematic of the time-domain voltage and current at the single port of the transmitting antenna.

$$V(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} v(t)e^{i\omega t}dt,\, I(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} i(t)e^{i\omega t}dt \quad (1a,b)$$

where ω is the angular frequency. Since $v(t)$ and $i(t)$ are real valued functions, the equations in (1) imply that

$$V(-\omega) = V^*(\omega), \quad I(-\omega) = I^*(\omega) \qquad (2a,b)$$

where the * denotes the complex conjugate.

Assume that the frequency domain antenna voltages and currents are bounded, bandlimited (they are zero or less than the noise level outside a finite frequency band), Riemann integrable functions of the angular frequency ω. Thus, the inverse Fourier transforms of $V(\omega)$ and $I(\omega)$ exist with their corresponding time-domain voltage and current

$$v(t) = \int_{-\infty}^{+\infty} V(\omega)e^{-i\omega t}d\omega, \quad i(t) = \int_{-\infty}^{+\infty} I(\omega)e^{-i\omega t}d\omega. \quad (3a,b)$$

In order for the $v(t)$ and $i(t)$ in equations (3) to equal those in (1) at every t, they should be restricted to continuous functions of t [5, p. 236]. However, the continuity of $v(t)$ and $i(t)$ need not be stated as an additional restriction because for a finite bandwidth with $V(\omega)$ and $I(\omega)$ bounded and Riemann integrable (and thus absolutely integrable [4, p. 116]), it follows that $v(t)$ and $i(t)$ are continuous [5, th. 2.7 on p. 225 with footnote on p. 222].

Strictly speaking, $V(\omega)$ and $I(\omega)$ cannot be perfectly bandlimited (that is, exactly zero outside a finite bandwidth, $|\omega| < \omega_0$) if $v(t)$ and $i(t)$ are assumed to be perfectly time-limited (that is, exactly zero outside a finite interval, $|t| < T_0$). Nonetheless, all known generators that feed antennas produce signals that decay into the noise levels beyond a given frequency and thus are effectively bandlimited. Of course, this effective bandwidth in the frequency-domain limits the narrowness of the pulse widths in the time domain signals. However, from a theoretical standpoint, this limitation is not prohibitive because an indefinitely large, but effectively finite frequency-domain bandwidth, can produce an indefinitely narrow, finite time-domain pulse width (see Footnote 2).

Also, one can assume that $v(t)$ and $i(t)$ are not perfectly but effectively time-limited (decaying into the noise levels beyond a finite time) without jeopardizing the finite bandwidth assumption for $V(\omega)$ and $I(\omega)$. In other words, the Fourier transform pairs $[v(t), i(t)]$ and $[V(\omega), I(\omega)]$ can both be effectively (but not perfectly) zero outside finite time and frequency domains. For example, the Fourier transform of an exponentially decaying Gaussian pulse $[e^{-t^2/(2\sigma^2)}]$ is another Gaussian function $[e^{-\sigma^2\omega^2/2}]$ that decays exponentially with frequency.

2.1 Definitions of Passivity and Causality

The transmitting antenna is *passive* if no net energy can be extracted at any time from the antenna as it is fed by the time-domain voltage or current. It is assumed that no energy is stored in the antenna before the voltage or current is applied. Mathematically, this "strong" passivity can be expressed as

$$\int_{-\infty}^{t} v(\tau)i(\tau)d\tau \geq 0 \qquad (4)$$

for all t and for all allowable $v(\tau)$ and $i(\tau)$. Since the allowable $v(\tau)$ and $i(\tau)$ are time-limited, bounded, and Riemann integrable, the Riemann integral for the energy in (4) is well defined (exists) [4, p. 116].

The single-port transmitting antenna is ***causal*** if for every allowable input current $i(\tau)$ or voltage $v(\tau)$ that is zero at the terminal of the input port for $\tau < t$, the output voltage $v(\tau)$ or current $i(\tau)$, respectively, at the same port terminal will also be zero for $\tau < t$.

The rest of the paper is devoted to showing that the passivity expressed in (4) for a linear, time invariant transmitting antenna implies that the antenna is causal.

3 Input Impedance for a Linear, Time-Invariant Transmitting Antenna

Linearity for the one-port, time-invariant transmitting antenna can be defined by the frequency-domain voltage $V(\omega)$ being equal to the frequency-domain current $I(\omega)$ multiplied by a frequency-domain input impedance $Z(\omega)$ at each frequency ω, that is

$$V(\omega) = Z(\omega)I(\omega). \qquad (5)$$

Combining (5) with (2), one finds that

$$Z(-\omega) = Z^*(\omega). \qquad (6)$$

Since it is assumed that $V(\omega)$ is a bounded function of ω, the input impedance $Z(\omega)$ must not have singularities over the effective operational bandwidth of $I(\omega)$. If at every frequency within its operational bandwidth, the antenna radiates a nonzero power (through a finite nonzero radiation resistance) when excited by a nonzero applied voltage $V(\omega)$, then $I(\omega)$ cannot be zero for this nonzero $V(\omega)$ and thus $Z(\omega)$ has to be bounded within the effective bandwidth of the antenna.[1]

However, unlike $V(\omega)$ and $I(\omega)$, we do not want to assume that $Z(\omega)$ is bandlimited because it should be general enough to represent antennas with network elements such as resistors, inductors, and capacitors having impedances R, $-i\omega L$, and $i/(\omega C)$, respectively. Therefore, in order to represent the time-domain input impedance as an inverse Fourier transform of $Z(\omega)$ that converges for all t, we insert an exponentially decaying factor into the inverse-transform integrand to obtain what has been called the "analytic-signal" transform

$$z_\alpha(t) = \int_{-\infty}^{+\infty} Z(\omega)e^{-i\omega t}e^{-|\omega|\alpha}d\omega \qquad (7)$$

where α is a positive real constant ($\alpha > 0$). With the help of (6), this "analytic-signal" transform in (7) can be rewritten as [6, sec. 5.3]

$$z_\alpha(t) = 2\int_{0}^{+\infty} \mathrm{Re}[Z(\omega)e^{-i\omega t}]e^{-\omega\alpha}d\omega \qquad (8)$$

where "Re" denotes the "real part". In this paper, we will work with (7) rather than (8) to take advantage of some useful theorems that apply to the full $\pm\infty$ integral in (7).

It is assumed that for any $\alpha > 0$, the function $Z(\omega)e^{-|\omega|\alpha}$ approaches zero fast and smooth enough as $|\omega| \to \infty$ that

[1] Many circuits have lossless poles (unbounded singularities) in $Z(\omega)$ but they do not represent realistic antennas that have some radiation loss and usually some dissipative loss throughout their frequency bandwidth. These losses reduce the infinite singularities of the lossless poles to finite values.

it does not prevent the convergence of Riemann integration (because of the exponential decay of $e^{-|\omega|\alpha}$). If, in addition, $Z(\omega)e^{-|\omega|\alpha}$ is a bounded, Riemann integrable function of ω (and thus absolutely Riemann integrable [4, pp. 115–116]) for $\alpha > 0$, then $z_\alpha(t)$ in (7) is a continuous function of t [5, th. 2.7 on p. 225 with footnote on p. 222] for $\alpha > 0$. The limit of the integral in (7) as $\alpha \to 0$ cannot always be taken for some values of t because for some $Z(\omega)$ the function $\lim_{\alpha\to 0} z_\alpha(t)$ can contain infinite singularities at some values of t and thus is not Riemann integrable. (For example, if $Z(\omega) = R$, then $\lim_{\alpha\to 0} z_\alpha(0) = \infty$.)

Nevertheless, we can take the limit as $\alpha \to 0$ to recover $Z(\omega)$ from the Fourier transform of $z_\alpha(t)$ in (7), namely

$$Z(\omega)e^{-|\omega|\alpha} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} z_\alpha(t)e^{i\omega t}dt \qquad (9)$$

so that

$$Z(\omega) = \lim_{\alpha\to 0}\left[Z(\omega)e^{-|\omega|\alpha}\right] = \frac{1}{2\pi}\lim_{\alpha\to 0}\int_{-\infty}^{+\infty} z_\alpha(t)e^{i\omega t}dt. \qquad (10)$$

A sufficient condition for (7) and (9) to be Fourier transform pairs is for $Z(\omega)e^{-|\omega|\alpha}$ to be a continuous, piecewise differentiable, absolutely integrable function of ω [7, theorem 4-5]. Alternatively, the condition of piecewise differentiability of $Z(\omega)e^{-|\omega|\alpha}$ can be replaced by the absolute integrability of $z_\alpha(t)$ [5, p. 236]. It can be proven that the absolute integrability of $z_\alpha(t)$ as $\alpha \to 0$ is a necessary and sufficient condition for the electrical stability of the antenna.

3.1 Input Impedance Convolution Integral

To obtain a convolution integral for $v(t)$ in terms of $z_\alpha(t)$ and $i(t)$, we use ($3a$) and (5) to write

$$v(t) = \int_{-\infty}^{+\infty} V(\omega)e^{-i\omega t}d\omega = \int_{-\infty}^{+\infty} Z(\omega)I(\omega)e^{-i\omega t}d\omega \qquad (11)$$

which can be rewritten as

$$v(t) = \int_{-\infty}^{+\infty} \lim_{\alpha\to 0}[Z(\omega)e^{-|\omega|\alpha}]I(\omega)e^{-i\omega t}d\omega. \qquad (12)$$

Since $I(\omega)$ and $V(\omega)$ are effectively bandlimited functions of ω, the $\pm\infty$ limits of integration can be made finite ($\pm\Omega$) so that there is negligible difference between $\int_{-\Omega}^{+\Omega} Z(\omega)e^{-|\omega|\alpha}I(\omega)e^{-i\omega t}d\omega$ and the same integral with $\pm\infty$ limits for all $|\alpha| < \alpha_0$. Then with $Z(\omega)e^{-|\omega|\alpha}$ a continuous bounded function of ω within $\pm\Omega$, the derivative with respect to α of this integral can be brought inside the integral sign [8, vol. II, p. 218], a result that implies that the α limit in (12) can be brought outside the integral to get

$$v(t) = \lim_{\alpha\to 0}\int_{-\infty}^{+\infty} [Z(\omega)e^{-|\omega|\alpha}]I(\omega)e^{-i\omega t}d\omega. \qquad (13)$$

Applying the convolution theorem to the integral in (13) along with the inverse Fourier transforms in ($3b$) and (7), one recasts (13) in the form

$$v(t) = \frac{1}{2\pi}\lim_{\alpha\to 0}\int_{-\infty}^{+\infty} z_\alpha(t-t')i(t')dt' \qquad (14)$$

which is a time-domain input-impedance convolution integral for $v(t)$ in terms of $i(t)$. Sufficient conditions for the convolution theorem to apply to the integral in (13) are that $I(\omega)$ and $Z(\omega)e^{-|\omega|\alpha}$ are continuous, absolutely integrable functions of ω for $\alpha > 0$, and that $i(t)$ is an absolutely integrable function of t [7, theorem 4-8(iv)].

4 Proof of Causality

Causality can be derived from passivity with the help of the convolution integral in (14). To do this, begin by choosing a time-domain input current function $i(t)$ in (14) that is zero for t less than some t_0, that is

$$i(t) = 0, \quad \text{for } t < t_0 \qquad (15)$$

and satisfies the passivity condition in (4). Next consider any allowable (finite duration, bounded, and Riemann integrable) current function $i_1(t)$ that satisfies the passivity condition in (4), and a second current function $i_2(t)$ related linearly to $i_1(t)$ and $i(t)$ by the simple sum

$$i_2(t) = i_1(t) + Ai(t) \qquad (16)$$

where A can be any real number (positive, negative, or zero). The current $i_2(t)$ satisfies the passivity condition in (4) because both $i(t)$ and $i_1(t)$ satisfy this passivity condition. Neither of the functions $i_1(t)$ or $i_2(t)$ are necessarily zero for $t < t_0$ However, in view of (15), we see that

$$i_2(t) = i_1(t), \quad \text{for } t < t_0. \qquad (17)$$

Substitution of $i_2(t)$ from (16) into the convolution integral of (14) gives

$$v_2(t) = v_1(t) + Av(t). \qquad (18)$$

From the definition of passivity in (4)

$$\int_{-\infty}^{t} v_2(\tau)i_2(\tau)d\tau = \int_{-\infty}^{t} [v_1(\tau) + Av(\tau)][i_1(\tau) + Ai(\tau)]d\tau \geq 0 \qquad (19)$$

for all t. For $t < t_0$, the identity in (15) or (17) shows that this expression reduces to

$$\int_{-\infty}^{t} [v_1(\tau) + Av(\tau)]i_1(\tau)d\tau \geq 0, \quad \text{for } t < t_0. \qquad (20)$$

Now assume the result contrary to what we want to prove, namely that $v(\tau)$ is nonzero for some interval during the time $\tau < t_0$. Then, since $i_1(\tau)$ can be any function within the allowable set of functions (finite duration, bounded, and Riemann integrable), there will be many functions

$i_1(\tau)$ for which

$$\int_{-\infty}^{t} v(\tau) i_1(\tau) d\tau \neq 0, \quad \text{for some } t < t_0. \tag{21}$$

This means that since A in (20) can be any real number with an unrestricted large positive or negative value, the integral in (20) can always be made less than zero for some $t < t_0$, therefore violating the inequality in (20). Consequently, we have a contradiction unless $v(\tau)$ is zero for all $\tau < t_0$.

In other words, beginning with any allowable input current $i(t)$, which is zero for all $t < t_0$ and satisfies the passivity condition in (4), the corresponding voltage $v(t)$ given by the linear impedance relation in (14) is also zero for all $t < t_0$; that is, passivity (4) implies causality.

Letting $i(t')$ in (14) approach the delta function $\delta(t' - t_0)$ (which is zero for $t' < t_0$) with respect to the test function $z_\alpha(t - t')$, we have $v(t) = z_\alpha(t - t_0)$ as $\alpha \to 0$.[2] Thus, for causal systems, $z_\alpha(t - t_0) = 0$ for $t < t_0$ as α approaches zero, that is, causality implies that the time-domain input-impedance function satisfies

$$\lim_{\alpha \to 0} z_\alpha(t) = 0, \quad t < 0. \tag{22}$$

Conversely if $z_\alpha(t)$ obeys (22), then (14) shows that the system is causal (that is, if $i(t) = 0$ for $t < t_0$, then $v(t) = 0$ for $t < t_0$). In all, the linear, time-invariant, one-port network or antenna is causal if and only if its time-domain input-impedance function is zero for all time less than zero as $\alpha \to 0$, that is, it obeys (22). This necessary and sufficient condition for causality is a property of all causal linear impulse response functions of which $z_\alpha(t)$ is a particular case [3].

We have shown that passivity implies causality and thus noncausality implies nonpassivity, but nonpassivity does not imply noncausality. For example, consider the impedance of a negative resistance $Z(\omega) = -R$ ($R > 0$), which is gainy (nonpassive). Taking the Fourier transform of this $Z(\omega)$ gives the corresponding $z_\alpha(t)$ in (7) that approaches $-2\pi R\delta(t)$ as $\alpha \to 0$ if used with test functions as in the convolution integral (14). Thus, $z_\alpha(t)$ is effectively causal as $\alpha \to 0$ because $\delta(t)$ is causal (that is, $\delta(t) = 0$ for $t < 0$). However, since $\int_{-\infty}^{t} v(\tau) i(\tau) d\tau = -2\pi R \int_{-\infty}^{t} i^2(\tau) d\tau < 0$, the system is nonpassive for all input currents.

5 Conclusion

Past proofs that passivity implies causality for linear, time-invariant networks represented by an input impedance have relied upon the advanced mathematical theorems of Lebesgue-measure linear operator theory and Schwartz's theory of distributions that are somewhat detached from the physics of these networks. Consequently, in this paper a relatively simple, straightforward, self-contained derivation is given, using only the mathematics of Riemann integration and parametric delta functions, to prove that the causality of single-port transmitting antennas with their voltages linearly related to their currents by an input impedance follows from the passivity of these antennas. Necessary and sufficient conditions for the causality of the antennas are given in terms of their time-domain input impedances. It will also be demonstrated in the talk using a parallel RLC circuit that if a frequency-dependent series resistance and reactance are used to represent the input impedance of an antiresonant passive antenna, neither the time-domain series resistance nor the time-domain series reactance is passive, even though the time-domain input impedance (their sum) satisfies the passivity, causality, and stability conditions [9]. Although all of the derivations are based on the input-impedance representation to define the linearity of the antennas, the analogous derivations hold as well for the input-admittance representation of antennas.

[2] One may object to letting $i(t)$ approach a Dirac delta function $\delta(t)$ because it has been assumed that $i(t)$ is a bounded, Riemann integrable function and that $I(\omega)$ is bandlimited. One can circumvent this problem by replacing $\delta(t)$ with a well-defined, parametric, finite delta-like function $\delta_n(t)$ (with a bandlimited frequency spectrum) that has the sifting property of the Dirac delta function as the parameter n becomes increasingly large. Because $z_\alpha(t)$ is a continuous function of t, this parametric delta-like function extracts $z_\alpha(t) + \varepsilon_n(t)$ where $|\varepsilon_n(t)|$ becomes increasingly small in value as the parameter n becomes increasingly large.

Acknowledgements

This research was supported in part under the U.S. Air Force Office of Scientific Research (AFOSR) Grant # FA9550-19-1-0097 through Dr. Arje Nachman.

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