URSI GASS 2021, Rome, Italy, 28 August - 4 September 2021

Accurate and Efficient Analysis of the Field Scattered from a Thin Dielectric Disk Near the Disk Natural Mode Resonances by means of GBC and MAP

M. Lucido* (1,2)
(1) Department of Electrical and Information Engineering "Maurizio Scarano" (DIEI), University of Cassino and Southern Lazio, 03043, Cassino, Italy
(2) ELEDIA Research Center (ELEDIA@UniCAS), University of Cassino and Southern Lazio, 03043, Cassino, Italy

Abstract

The generalized boundary conditions allow to formulate the problem of the electromagnetic scattering from a thin dielectric disk in terms of two decoupled surface integral equations for the effective electric and magnetic currents. Taking advantage of the revolution symmetry of the problem, such equations can be reduced to two infinite set of one-dimensional independent integral equations in the vector Hankel transform domain. A suitable analytical preconditioning procedure, based on Helmholtz decomposition and Galerkin method with a complete set of orthogonal eigenfunctions of the static part of the integral operator, reconstructing the physical behavior of the fields, as expansion basis, leads to fast converging Fredholm second-kind matrix equations even near the natural resonances of the disk.

1 Introduction

Dielectric disks are used as building blocks of many devices across a wide spectrum of frequencies mainly because they can work like open resonators, which can support even high-Q natural modes. The analysis of the electromagnetic scattering from such objects is frequently carried out by means of integral equation formulations because the radiation condition is taken into account by the choice of the Green's function of the problem and the unknowns are defined on finite supports. However, a rigorous full-wave analysis of a disk with a finite thickness usually requires a rather complicated vector formulation to be numerically solved. Moreover, the discretization needed near the natural resonances of the disk can be particularly burdensome in terms of memory requirement due to the great variability of the fields for small changes of the frequency or the material parameters. When dealing with a thin dielectric disk, we can imagine considering a zero-thickness disk providing generalized boundary conditions (GBC) [1-3] on the disk surface and guaranteeing the local power boundedness by means of suitable edge condition [4]. In this way, the problem can be formulated in terms of two decoupled surface integral equations for the effective electric and magnetic currents, respectively. The formulation can be further simplified by taking advantage of the revolution symmetry of the problem. Indeed, the Fourier series expansion of the fields combined with the vector Hankel transform allow to reduce the surface integral equations to two infinite sets of one-dimensional independent integral equations in the spectral domain.

A key point is the proper selection of the discretization scheme to be used due to the hypersingular nature of the obtained integral equations, for which the existence of a solution cannot be established and, if such a solution exists, the convergence of a discretization scheme cannot be generally stated.

The method of analytical preconditioning (MAP) [5], which makes simultaneously the discretization and the analytical regularization of an integral equation, allows to overcome this problem. Indeed, the obtained matrix equation is the sum of a continuously invertible operator and a completely continuous operator at which the Fredholm theory can be applied [6]. On the other hand, fast convergence is achieved by means of a suitable choice of the expansion basis, as demonstrated in a wide range of applications [7-15].

In this paper, the one-dimensional integral equation for the n-th harmonic of the electric/magnetic current is discretized by means of Helmholtz decomposition and Galerkin method with a complete set of orthogonal eigenfunctions of the static part of the integral operator, reconstructing the physical behavior of the fields around the center and at the edge of the disk. In this way, a fast converging Fredholm second-kind matrix operator equation is obtained. The proposed numerical results show the effectiveness of the technique detailed above in reconstructing the solution even near the natural resonances of the disk.

2 Formulation of the Problem and Proposed Solution

A thin dielectric disk of radius a and thickness τ is immersed in free space. A cylindrical coordinate system (ρ, ϕ, z) with the origin at the center of the disk and the z axis orthogonal to it is introduced. A plane wave, $(\underline{E}^{inc}, \underline{H}^{inc})$, impinges onto the disk surface generating a scattered field, $(\underline{E}^{sc}, \underline{H}^{sc})$, such that the total field, $(\underline{E}, \underline{H})$, is given by the sum of the incident field and the scattered field. Supposing $\tau \ll \lambda$ and $\tau \ll a$, where λ denotes the free space wavelength, the problem can be formulated in terms of two surface integral equations by imposing the GBC on the median surface of the disk, located at $z = 0$, i.e., [1]

$$\hat{z} \times \left(\underline{E}\big|_{z=0^+} + \underline{E}\big|_{z=0^-} \right) \times \hat{z} = 2R_e \underline{J}_e \tag{1a}$$

$$\hat{z} \times \left(\underline{H}\big|_{z=0^+} + \underline{H}\big|_{z=0^-} \right) \times \hat{z} = 2R_m \underline{J}_m \tag{1b}$$

for $\rho \leq a$, where the effective electric and magnetic currents are the jumps across the median surface,

$$\underline{J}_e = \hat{z} \times \left(\underline{H}^{sc}\big|_{z=0^+} - \underline{H}^{sc}\big|_{z=0^-} \right), \tag{2a}$$

$$\underline{J}_m = -\hat{z} \times \left(\underline{E}^{sc}\big|_{z=0^+} - \underline{E}^{sc}\big|_{z=0^-} \right), \tag{2b}$$

while R_e and R_m are the electric and magnetic resistivities of the disk, respectively. It is interesting to observe that the equations (1) decouple for a planar surface. Hence, equation (1a) is for the effective electric current, while equation (1b) is for the magnetic one. Moreover, the unique solvability of the problem in hand is guaranteed by the boundary conditions, the edge condition and the radiation condition [5].

Due to the revolution symmetry of the problem, the two surface integral equations in the spatial domain can be equivalently reduced to two infinite sets of independent one-dimensional integral equations in the Hankel transform domain by means of the Fourier series expansion of the fields. The equation for the n-th harmonic of the electric/magnetic current is given by [16]

$$\int_0^{+\infty} \underline{\underline{\mathbf{H}}}^{(n)}(w\rho) \left[\underline{\underline{\tilde{\mathbf{G}}}}_r(w) - R_r \underline{\underline{I}} \right] \underline{\tilde{\mathbf{J}}}_r^{(n)}(w)\, w\, dw = -\underline{\mathbf{F}}_r^{inc(n)}(\rho, 0) \tag{3}$$

for $\rho \leq a$, $r = e, m$, $F_e = E$, $F_m = H$, where the symbol

$$\underline{\mathbf{P}}^{(n)}(\cdot) = \begin{pmatrix} P_\rho^{(n)}(\cdot) \\ -jP_\phi^{(n)}(\cdot) \end{pmatrix} \tag{4}$$

has been introduced, the kernel $\underline{\underline{\mathbf{H}}}^{(n)}(w\rho)$ and the spectral domain Green's functions $\underline{\underline{\tilde{\mathbf{G}}}}_r(w)$ have been defined in [16], and $\underline{\tilde{\mathbf{J}}}_r^{(n)}(w)$ is the vector Hankel transform of order n (VHT$_n$) of the n-th harmonic of the electric/magnetic current, $\underline{\mathbf{J}}_r^{(n)}(\rho)$.

By means of Helmholtz decomposition [17], each unknown can be represented as the superposition of a surface curl-free contribution, $\underline{\mathbf{J}}_{r,C}^{(n)}(\rho)$, and a surface divergence free-contribution, $\underline{\mathbf{J}}_{r,D}^{(n)}(\rho)$. It is simple to demonstrate that the VHT$_n$ of these contributions have only one nonvanishing component, i.e.,

$$\underline{\tilde{\mathbf{J}}}_{r,C}^{(n)}(w) = \begin{pmatrix} \tilde{J}_{r,C}^{(n)}(w) \\ 0 \end{pmatrix}, \tag{5a}$$

$$\underline{\tilde{\mathbf{J}}}_{r,D}^{(n)}(w) = \begin{pmatrix} 0 \\ -j\tilde{J}_{r,D}^{(n)}(w) \end{pmatrix}. \tag{5b}$$

Therefore, assuming $\underline{\mathbf{J}}_{r,T}^{(n)}(\rho)$ with $T = C, D$ as new unknowns in the spatial domain, scalar unknowns in the spectral domain can be handled.

The general integral equation in (3) is discretized by adopting the analytical preconditioning procedure presented in [18-20] and generalized in [21], which leads to a Fredholm second-kind matrix operator equation. It is based on Galerkin discretization scheme combined with a proper selection of the expansion functions. In our case, suitable expansion series in the spectral domain are the following [22]:

$$\tilde{J}_{r,T}^{(n)}(w) = \sum_{h=-1+\delta_{n,0}}^{+\infty} \gamma_{r,T,h}^{(n)} \sqrt{2\eta_{T,h}^{(n)}} \frac{J_{\eta_{T,h}^{(n)}}(aw)}{w^{p_T}}, \tag{6}$$

where $\delta_{n,m}$ is the Kronecker delta and $\gamma_{r,T,h}^{(n)}$ denotes the general expansion coefficient, $J_n(\cdot)$ is the Bessel function of the first kind and order n [23], $\eta_{T,h}^{(n)} = |n| + 2h + p_T + 1$, $p_C = 3/2$ and $p_D = 1$. It is interesting to note that, the functions in (6) constitute a complete set of orthonormal eigenfunctions of the static part of the integral operator, which reconstruct the physical behaviour of the components of the n-th harmonic of the currents around the center of the disk and at the edge. As a result, the guaranteed convergence is even fast, i.e., few expansion functions reconstruct the unknowns with a good accuracy. Moreover, the coefficient matrix elements are one-dimensional improper integrals efficiently evaluated by means of the analytical procedure developed in [20, 21].

3 Numerical Results

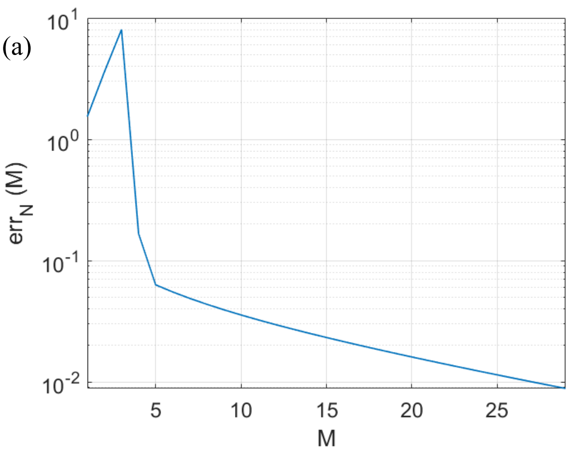
An approximate solution can be obtained by truncating the obtained infinite matrix equations. The convergence rate of the proposed method can be shown by introducing the following normalized truncation error:

$$\mathrm{err}_{r,N}(M) = \sqrt{\sum_{n=-N+1}^{N-1} \left\| \mathbf{x}_{r,M+1}^{(n)} - \mathbf{x}_{r,M}^{(n)} \right\|^2 \Big/ \sum_{n=-N+1}^{N-1} \left\| \mathbf{x}_{r,M}^{(n)} \right\|^2}, \tag{7}$$

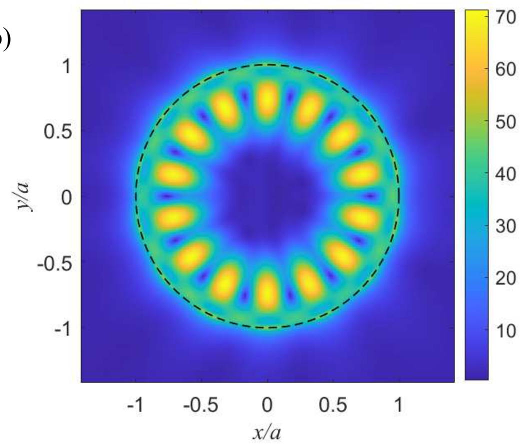
where $2N - 1$ is the number of the considered harmonics estimated as in [24], $\|\cdot\|$ is the usual Euclidean norm and $\mathbf{x}_{r,M}^{(n)}$ is the vector of all the expansion coefficients of the n-th harmonic of the electric/magnetic current evaluated using M expansion functions for each unknown.

Figure 1a shows $\mathrm{err}_N(M) = \max\{\mathrm{err}_{e,N}(M), \mathrm{err}_{m,N}(M)\}$ for the dielectric disk with $\varepsilon_r = 1000 - j10^{-2}$ (this unusual value can be associated with one of the novel colossalpermittivity materials [25]) and $\tau/a = 0.001$ near a whispering-gallery mode (WGM) resonance, i.e., for $a/\lambda = 0.71707$, when a TE polarized plane wave with $\left|\underline{E}^{inc}\right| = 1\text{V/m}$ impinges onto the disk at grazing incidence. The overall number of harmonics used is $2N-1=25$. Moreover, an error below 10^{-1} is obtained for $M=5$, while $M=28$ allows to achieve an error below 10^{-2}. In the last case, about 40 seconds are needed to fill the coefficient matrix on a laptop equipped with an Intel Core I7-10510U 1.80GHz–2.30GHz, 16GB RAM, running Windows 10 Home 64bit by means of an adaptive Gaussian quadrature routine. Hence, the convergence is very fast in terms of both computation time and storage requirement. In figure 1b, the near E-field behaviour in the disk plane shows the classical necklace pattern of the field hot spots and a bright edge due to the singularity of the fields at the edge of the considered model. To conclude, the bistatic radar cross-section (BRCS) in the disk plane plotted in figure 1c shows, as expected, the shadow lobe in the forward direction and intensive sidelobes around the disk.

(a)



(b)



(c)

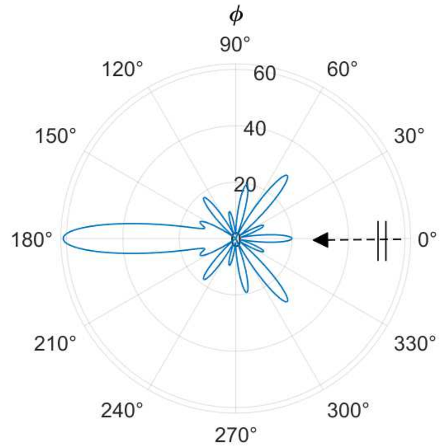


Figure 1. (a) Normalized truncation error, (b) Near E-field behavior, and (c) BRCS for the dielectric disk with $\varepsilon_r = 1000 - j10^{-2}$, $a/\lambda = 0.71707$, $\tau/a = 0.001$, when a TE polarized plane wave with $\left|\underline{E}^{inc}\right| = 1\text{V/m}$ impinges onto the disk at grazing incidence.

6 Acknowledgements

This research was supported in part by MIUR program “Dipartimenti di Eccellenza 2018-2022”.

7 References

1. E. Bleszynski, M. Bleszynski, T. Jaroszewicz, “Surface-integral equations for electrmagnetic scattering from impenetrable and penetrable sheets,” *IEEE Antennas Propag. Magazine*, **35**, 1993, pp. 14-24.

2. Z. Nazarchuk, K. Kobayashi, “Mathematical modelling of electromagnetic scattering from a thin penetrable target,” *Prog. Electromagn. Res.*, **55**, 2005, pp. 95-116.

3. M. V. Balaban, R. Sauleau, T. M. Benson, A. I. Nosich, “Dual integral equations technique in electromagnetic wave scattering by a thin disk,” *Prog. Electromagn. Res. B*, **16**, 2009, pp. 107-126.

4. D. S. Jones, *The theory of electromagnetism*, Pergamon Press, New York, NY, USA, 1964.

5. A. I. Nosich, “Method of analytical regularization in computational photonics,” *Radio Sci.*, **51**, 2016, pp. 1421-1430.

6. L. V. Kantorovich, G. P. Akilov, *Functional Analysis, 2nd ed.*, Pergamon Press, Oxford-Elmsford, NY, USA, 1982.

7. E. I. Veliev, V. V. Veremey, “Numerical-analytical approach for the solution to the wave scattering by polygonal cylinders and flat strip structures,” in *Analytical and Numerical Methods in Electromagnetic*

Wave Theory, H. Hashimoto, M. Idemen, O. A. Tretyakov, Eds., Science House, Tokyo, Japan, 1993.

8. N. Y. Bliznyuk, A. I. Nosich, A. N. Khizhnyak, “Accurate computation of a circular-disk printed antenna axisymmetrically excited by an electric dipole,” *Microw. Opt. Technol. Lett.*, **25**, 2000, pp. 211-216.

9. V. Losada, R. R. Boix, F. Medina, “Fast and accurate algorithm for the short-pulse electromagnetic scattering from conducting circular plates buried inside a lossy dispersive halfspace.” *IEEE Trans. Geosci. Remote Sens.*, **41**, 2003, pp. 988-997.

10. K. Hongo, Q. A. Naqvi, “Diffraction of electromagnetic wave by disk and circular hole in a perfectly conducting plane,” *Prog. Electromagn. Res.*, **68**, 2007, pp. 113-150.

11. M. Lucido, “Electromagnetic scattering by a perfectly conducting rectangular plate buried in a lossy half-space,” *IEEE Trans. Geosci. Remote Sens.*, **52**, 2014, pp. 6368-6378.

12. M. Lucido, “Scattering by a tilted strip buried in a lossy half-space at oblique incidence,” *Prog. Electromagn. Res. M*, **37**, 2014, pp. 51-62.

13. F. Corsetti, M. Lucido, and G. Panariello, “Effective analysis of the propagation in coupled rectangular-core waveguides,” *IEEE Photon. Technol. Lett.*, **26**, 2014, pp. 1855-1858.

14. M. Lucido, F. Schettino, M. D. Migliore, D. Pinchera, F. Di Murro, and G. Panariello, “Electromagnetic scattering by a zero-thickness PEC annular ring: a new highly efficient MoM solution,” *J. Electromagn. Waves Appl.*, Vol. 31, No. 4, pp. 405-416, 2017.

15. M. Lucido, C. Santomassimo and G. Panariello, “The method of analytical preconditioning in the analysis of the propagation in dielectric waveguides with wedges,” *J. Light. Technol.*, **36**, 2018, pp. 2925-2932.

16. M. Lucido, M. V. Balaban, S. V. Dukhopelnykov, A. I. Nosich, “A fast-converging scheme for the electromagnetic scattering from a thin dielectric disk,” *Electronics*, **9**, 2020, art. no 1451.

17. J. Van Bladel, “A discussion of Helmholtz’ theorem on a surface,” *AEÜ*, **47**, 1993, pp. 131-136.

18. M. Lucido, M. D. Migliore, D. Pinchera, “A new analytically regularizing method for the analysis of the scattering by a hollow finite-length PEC circular cylinder,” *Prog. Electromagn. Res. B*, **70**, 2016, pp. 55-71.

19. M. Lucido, G. Panariello, F. Schettino, “Scattering by a zero-thickness PEC disk: a new analytically regularizing procedure based on Helmholtz decomposition and Galerkin method,” *Radio Sci.*, **52**, 2017, pp. 2-14.

20. M. Lucido, F. Di Murro, G. Panariello, “Electromagnetic scattering from a zero-thickness PEC disk: A note on the Helmholtz-Galerkin analytically regularizing procedure,” *Progr. Electromagn. Res. Lett.*, **71**, 2017, pp. 7-13.

21. M. Lucido, F. Schettino, G. Panariello, “Scattering from a thin resistive disk: A guaranteed fast convergence technique,” *IEEE Trans. Antennas Propag.*, **69**, 2021, pp. 387-396.

22. J. E. Wilkins, “Neumann series of Bessel functions,” *Trans. Amer. Math. Soc.*, **64**, 1948, pp. 359-385.

23. M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, Verlag Harri Deutsch, Frankfurt, Germany, 1984.

24. N. Geng, L. Carin, “Wide-band electromagnetic scattering from a dielectric BOR buried in a layered lossy dispersive medium,” *IEEE Trans. Antennas Propag.*, **47**, 1999, pp. 610-619.

25. S. Krohns, *et al.*, “Colossal dielectric constant up to gigahertz at room temperature,” *Appl. Phys. Lett.*, **94**, 2009, pp. 2903-2905.