



Radiation of Electromagnetic Waves with Helical Phase Fronts from Nonsymmetric Sources Immersed in a Magnetoplasma

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Abstract

The radiation efficiency of antennas capable of exciting electromagnetic waves with helical phase fronts in a homogeneous magnetoplasma is studied. The emphasis is placed on calculating the power radiated from a nonsymmetric source in the form of two appropriately phased, crossed loop antennas using an approach that is based on an eigenfunction expansion representation of the excited field. Analytical and numerical results are reported for such a source in the nonresonant part of the whistler range.

1 Introduction

Sources that are capable of exciting electromagnetic waves with helical phase fronts in a magnetoplasma have recently been discussed in some detail, in particular with application to the whistler frequency range (see, e.g., [1–3] and references therein). Such waves propagate both axially and azimuthally with respect to a static magnetic field superimposed on the plasma. As a result, they carry orbital angular momentum along with the spin angular momentum. Although the excitation of whistler waves with helical phase fronts has already been studied experimentally in large plasma devices [1–3], no theoretical discussion of the radiation efficiency of the sources of such waves exists in the literature. In the present work, this issue will be addressed using a rigorous approach that is based on an eigenfunction expansion representation of the excited field [4, 5].

2 Basic Formulation

We consider a homogeneous cold magnetoplasma with an external static magnetic field \mathbf{B}_0 that is aligned with the z axis of a cylindrical coordinate system (ρ, ϕ, z) . The plasma medium is described by the dielectric tensor

$$\hat{\epsilon} = \epsilon_0 \begin{pmatrix} \epsilon & -ig & 0 \\ ig & \epsilon & 0 \\ 0 & 0 & \eta \end{pmatrix}, \quad (1)$$

where ϵ_0 is the permittivity of free space and the elements ϵ , g , and η can be found elsewhere [5]. Let the field in such a medium be excited by a time-harmonic electric current whose density $\mathbf{J}(\mathbf{r})$, with $\exp(i\omega t)$ time dependence dropped, is nonzero in a certain spatially limited region.

As is known [4, 5], the transverse (to \mathbf{B}_0) field components in the source-free regions of a magnetoplasma can be expressed in terms of the longitudinal components E_z and H_z , which are represented as

$$\begin{bmatrix} E_{z;m,s,\alpha}(\mathbf{r}, q) \\ H_{z;m,s,\alpha}(\mathbf{r}, q) \end{bmatrix} = \begin{bmatrix} E_{z;m,s,\alpha}(\rho, q) \\ H_{z;m,s,\alpha}(\rho, q) \end{bmatrix} \times \exp[-im\phi - ik_0 p_{s,\alpha}(q)z], \quad (2)$$

where q is the transverse wave number normalized to the free-space wave number k_0 , m is the azimuthal index ($m = 0, \pm 1, \pm 2, \dots$), the functions $p_{s,\alpha}(q)$ describe the dependences of the normalized (to k_0) longitudinal wave number p on the transverse wave number q for the “ordinary” ($\alpha = o$) and “extraordinary” ($\alpha = e$) normal waves of the plasma, the subscript s denotes the wave propagation direction ($s = +$ and $s = -$ correspond to the waves transferring energy in the positive and negative directions of the z axis, respectively), and $E_{z;m,s,\alpha}(\rho, q)$ and $H_{z;m,s,\alpha}(\rho, q)$ are functions describing the radial distributions of the longitudinal field components of a wave with the transverse wave number q and the indices m , s , and α . The functions $p_{s,\alpha}(q)$ satisfy the relationship $p_{+,\alpha}(q) \equiv p_\alpha(q) = -p_{-,\alpha}(q)$, where

$$p_\alpha(q) = \left[\epsilon - \frac{1}{2} \left(1 + \frac{\epsilon}{\eta} \right) q^2 + \chi_\alpha R(q) \right]^{1/2},$$

$$R(q) = \left[\frac{1}{4} \left(1 - \frac{\epsilon}{\eta} \right)^2 q^4 - \frac{g^2}{\eta} q^2 + g^2 \right]^{1/2}. \quad (3)$$

Here, $\chi_o = -\chi_e = -1$ and it is assumed that $\text{Im}[p_\alpha(q)] < 0$ and $\text{Re}[R(q)] > 0$ (see [5] for more details).

As is shown in [4, 5], the total source-excited field in a homogeneous magnetoplasma can always be represented by expansion in terms of the vector modal fields with purely real positive values of q and the longitudinal field components which are written as

$$E_{z;m,s,\alpha}(\rho, q) = i^{m+1} \eta^{-1} n_{s,\alpha} q J_m(k_0 q \rho),$$

$$H_{z;m,s,\alpha}(\rho, q) = -Z_0^{-1} i^m q J_m(k_0 q \rho), \quad (4)$$

where Z_0 is the free-space impedance, J_m is the Bessel function of the first kind of order m , and

$$n_{s,\alpha}(q) = -\epsilon [q^2 + p_\alpha^2(q) + g^2/\epsilon - \epsilon] [g p_{s,\alpha}(q)]^{-1}. \quad (5)$$

Other field components corresponding to (4) are written as

$$\begin{aligned} E_{\rho;m,s,\alpha}(\rho,q) &= i^m \left[(1+u_\alpha) J_{m+1}(k_0 q \rho) - u_\alpha m \frac{J_m(k_0 q \rho)}{k_0 q \rho} \right], \\ E_{\phi;m,s,\alpha}(\rho,q) &= i^{m+1} \left[J_{m+1}(k_0 q \rho) + u_\alpha m \frac{J_m(k_0 q \rho)}{k_0 q \rho} \right], \\ H_{\rho;m,s,\alpha}(\rho,q) &= \frac{i^{m-1}}{Z_0} \left[p_{s,\alpha} J_{m+1}(k_0 q \rho) - v_{s,\alpha} m \frac{J_m(k_0 q \rho)}{k_0 q \rho} \right], \\ H_{\phi;m,s,\alpha}(\rho,q) &= \frac{i^{m+2}}{Z_0} \left[n_{s,\alpha} J_{m+1}(k_0 q \rho) - v_{s,\alpha} m \frac{J_m(k_0 q \rho)}{k_0 q \rho} \right], \end{aligned} \quad (6)$$

where

$$u_\alpha = g^{-1} [q^2 + p_\alpha^2(q) - \varepsilon] - 1, \quad v_{s,\alpha} = p_{s,\alpha}(q) + n_{s,\alpha}(q). \quad (7)$$

Upon multiplying the functions in (6) by the same exponential function as in (2), we obtain the transverse components of the electric and magnetic fields $\mathbf{E}_{m,s,\alpha}(\mathbf{r},q)$ and $\mathbf{H}_{m,s,\alpha}(\mathbf{r},q)$ of modes in terms of which the source-excited field can be expanded.

According to [4, 5], the field in the source-free regions is represented as

$$\begin{bmatrix} \mathbf{E}(\mathbf{r}) \\ \mathbf{H}(\mathbf{r}) \end{bmatrix} = \sum_{m=-\infty}^{\infty} \sum_{\alpha} \int_0^{\infty} a_{m,s,\alpha}(q) \begin{bmatrix} \mathbf{E}_{m,s,\alpha}(\mathbf{r},q) \\ \mathbf{H}_{m,s,\alpha}(\mathbf{r},q) \end{bmatrix} dq, \quad (8)$$

where the waves transferring energy in either the positive or negative direction of the z axis, respectively, are used for the total field on different sides of the source region relative to this axis, and the expansion coefficients $a_{m,s,\alpha}$ are given by the formula

$$a_{m,\pm,\alpha}(q) = \frac{1}{N_{m,\alpha}(q)} \int_V \mathbf{J}(\mathbf{r}) \cdot \mathbf{E}_{-m,\mp,\alpha}^{(T)}(\mathbf{r},q) d\mathbf{r}. \quad (9)$$

Here, integration is performed over the volume V occupied by a source with the current density $\mathbf{J}(\mathbf{r})$, the superscript (T) denotes the field taken in an auxiliary (“transposed”) medium described by the transposed dielectric tensor $\hat{\varepsilon}^T$, and the normalization quantity $N_{m,\alpha}(q)$ is represented as

$$N_{m,\alpha}(q) = 4\pi(-1)^{m+1} [1 + \eta^{-1} n_{s,\alpha}^2(q)] / [Z_0 k_0^2 p'_\alpha(q)], \quad (10)$$

where the prime indicates the derivative with respect to the argument. We do not present the field expansion inside the source region for the sake of brevity.

3 Power Radiated

The above results allow one to find the total power P radiated from the source by integrating the time-averaged Poynting vector \mathbf{S} over two infinite cross-sectional areas S_1 and S_2 which are located on the left and right (relative to the z -axis direction) of the source region, respectively. Thus, we have

$$P = \int_{S_1} \mathbf{S} \cdot (-\mathbf{z}_0) dS_{\perp} + \int_{S_2} \mathbf{S} \cdot \mathbf{z}_0 dS_{\perp}, \quad (11)$$

where \mathbf{z}_0 is the unit vector along the z axis. Evaluation of the integrals in (11) is significantly facilitated if we use the following power orthogonality relation, which is valid for propagating waves in a lossless magnetoplasma:

$$\begin{aligned} &\int_0^{2\pi} d\phi \int_0^{\infty} [\mathbf{E}_{m,s,\alpha}(\mathbf{r},q) \times \mathbf{H}_{\bar{m},\bar{s},\bar{\alpha}}^*(\mathbf{r},\tilde{q}) \\ &\quad + \mathbf{E}_{\bar{m},\bar{s},\bar{\alpha}}^*(\mathbf{r},\tilde{q}) \times \mathbf{H}_{m,s,\alpha}(\mathbf{r},q)] \cdot \mathbf{z}_0 \rho d\rho \\ &= 4\mathcal{P}_{m,s,\alpha}(q) \delta(q - \tilde{q}) \delta_{m,\bar{m}} \delta_{s,\bar{s}} \delta_{\alpha,\bar{\alpha}}. \end{aligned} \quad (12)$$

Here, $\delta(q)$ is the Dirac function, $\delta_{\alpha,\beta}$ is the Kronecker delta, and the asterisk denotes complex conjugation. The normalization quantities $\mathcal{P}_{m,+,\alpha}(q)$ and $\mathcal{P}_{m,-,\alpha}(q)$ for waves transferring energy in the positive and negative directions of the z axis, respectively, obey the relation $\mathcal{P}_{m,+,\alpha}(q) = -\mathcal{P}_{m,-,\alpha}(q) = \mathcal{P}_{m,\alpha}(q)$, where $\mathcal{P}_{m,\alpha}(q) = (-1)^m N_{m,\alpha}(q)/4$. A proof of (12) can be obtained using a technique similar to that discussed in [4].

With allowance for (8)–(12), the total radiated power takes the following form:

$$P = \sum_{m=-\infty}^{\infty} \sum_{s=-}^{+} \sum_{\alpha} \int |a_{m,s,\alpha}(q)|^2 \mathcal{P}_{m,\alpha}(q) dq, \quad (13)$$

where the integration is performed over the positive real values of q for which the longitudinal wave numbers $p_{s,\alpha}(q)$ with $\alpha = o$ and $\alpha = e$ are purely real.

4 Radiation from a Source with the Rotating Near-Zone Magnetic Field

We now apply the developed approach to the radiation from a source capable of exciting waves with helical phase fronts in a magnetoplasma. The simplest source of this kind is that with a rotating near-zone magnetic field. Such a source can be realized using two loops of radius a with perpendicular axes and a $(\mp\pi/2)$ delay in phase of their currents having the same magnitude I_0 [1–3], as is shown in Fig. 1.

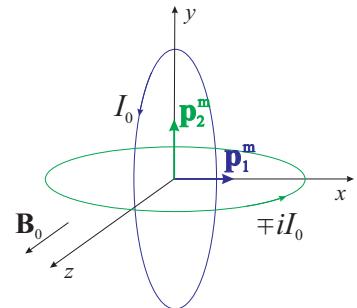


Figure 1. Geometry of the source.

Depending on the sign of the phase delay, the current density of the specified source can be written as $\mathbf{J}(\mathbf{r}) = \mathbf{J}_{\mp}(\mathbf{r}) = \mathbf{J}^{(1)}(\mathbf{r}) \mp i\mathbf{J}^{(2)}(\mathbf{r})$, where

$$\mathbf{J}^{(1)}(\mathbf{r}) = I_0 (-zy_0 + yz_0) a^{-1} \delta(x) \delta(\sqrt{y^2 + z^2} - a), \quad (14)$$

$$\mathbf{J}^{(2)}(\mathbf{r}) = I_0 (zx_0 - xy_0) a^{-1} \delta(y) \delta(\sqrt{x^2 + z^2} - a). \quad (15)$$

Here, \mathbf{x}_0 and \mathbf{y}_0 are the unit vectors aligned with the x and y axes of a Cartesian coordinate system, respectively. It is evident that the currents specified by (14) and (15) correspond to two magnetic dipoles that are parallel to the respective axes. Hence, the current distributions $\mathbf{J}_-(\mathbf{r})$ and $\mathbf{J}_+(\mathbf{r})$ describe a source with the right- or left-handed rotation of the equivalent magnetic dipole and, correspondingly, its near-zone magnetic field around the z axis.

Substituting the source current density into (9) and allowing for (14), (15), and the relationships

$$\begin{aligned} E_{\rho;-m,-s,\alpha}^{(T)}(\rho, q) &= -E_{\rho;m,s,\alpha}(\rho, q), \\ E_{\phi,z;-m,-s,\alpha}^{(T)}(\rho, q) &= E_{\phi,z;m,s,\alpha}(\rho, q), \end{aligned} \quad (16)$$

after some lengthy algebra one obtains

$$\begin{aligned} a_{m,s,\alpha}(q) = -\frac{I_0}{N_{m,\alpha}(q)} i8\delta_m \int_0^a &\left\{ \sin\left(k_0 p_{s,\alpha} \sqrt{a^2 - \rho^2}\right) \right. \\ \times \left[(1+u_\alpha) J_{m+1}(k_0 q \rho) - u_\alpha m \frac{J_m(k_0 q \rho)}{k_0 q \rho} \right] + \frac{qn_{s,\alpha}(q)}{\eta} \\ \times \cos\left(k_0 p_{s,\alpha} \sqrt{a^2 - \rho^2}\right) \frac{\rho J_m(k_0 q \rho)}{\sqrt{a^2 - \rho^2}} \Bigg\} d\rho. \end{aligned} \quad (17)$$

Here, the value of δ_m is determined by the rotation direction of the magnetic dipole moment of the source. For the source $\mathbf{J}_-(\mathbf{r})$ corresponding to the right-handed or clockwise rotation of this dipole, $\delta_m = 1$ if $m = 1 \pm 4M$ (hereafter, $M = 0, 1, 2, \dots$), and is zero otherwise. For the source $\mathbf{J}_+(\mathbf{r})$ corresponding to the left-handed or counterclockwise rotation, $\delta_m = 1$ if $m = -1 \pm 4M$, and is zero otherwise.

We now calculate the power radiated from the nonsymmetric sources in the nonresonant interval of the whistler frequency range, which is of great interest for many applications [1]. This interval is defined as

$$\Omega_c \ll \omega < \omega_{\text{LH}} \ll \omega_c < \omega_p, \quad (18)$$

where Ω_c is the ion cyclotron frequency, ω_{LH} is the lower hybrid frequency, and ω_c and ω_p are the cyclotron and plasma frequencies of electrons, respectively. In the frequency range (18), the elements of the dielectric tensor (1) can be written, to a good approximation, as

$$\varepsilon = \left(1 + \frac{\omega_p^2}{\omega_c^2}\right) \left(1 - \frac{\omega_{\text{LH}}^2}{\omega^2}\right), \quad g = -\frac{\omega_p^2}{\omega_c \omega}, \quad \eta = -\frac{\omega_p^2}{\omega^2}. \quad (19)$$

Recall that at the chosen frequencies, the “ordinary” wave is evanescent and cannot therefore contribute to the radiated power. At the same time, the “extraordinary” or whistler-mode wave is propagating when $0 < q < q_{\max}$, where $q_{\max} = [(\varepsilon^2 - g^2)/\varepsilon]^{1/2}$ [5]. Bearing this in mind and making use of the fact that the magnitudes of the quantities $a_{m,s,\alpha}(q)$ in (17) are independent of s for the considered sources, we can rewrite the general formula (13) as

$$P = \sum_{m=-\infty}^{\infty} P_m = \sum_{m=-\infty}^{\infty} 2 \int_0^{q_{\max}} |a_{m,+,\alpha}(q)|^2 \mathcal{P}_{m,\alpha}(q) dq, \quad (20)$$

where P_m is the partial power radiated to the m th azimuthal field harmonic with the phase $\omega t - m\phi - k_0 p_e z$. Note that due to the relationship $|a_{m,+,\alpha}(q)| = |a_{m,-,\alpha}(q)|$, each quantity P_m in (20) comprises two identical powers contributing to the radiation to both sides of the source, i.e., along the positive and negative directions of the z axis.

Generally, the expansion coefficients $a_{m,+,\alpha}(q)$ and the partial powers P_m can be calculated only numerically. In the frequency range (18), these quantities admit analytical representation in the case of an electrically small source where $k_0 q_{\max} a \ll 1$. Using the additional inequality $k_0 p_e a \ll 1$, which is also fulfilled in this case, one can replace the trigonometric and cylindrical functions in the integrand of (17) by their small-argument approximations. Then it turns out that only the lowest excited azimuthal harmonic, which corresponds to $M = 0$, gives the predominant contribution to the field and the radiated power of the considered sources. This means that for the current \mathbf{J}_- , it is sufficient to allow for the contribution only from the $m = 1$ harmonic with the expansion coefficient

$$a_{1,s,\alpha}(q) = -ik_0 I_0 \pi a^2 [p_{s,\alpha}(q) + n_{s,\alpha}(q)] / N_{1,\alpha}(q). \quad (21)$$

For the current \mathbf{J}_+ , the predominant contribution comes from the $m = -1$ harmonic with the expansion coefficient

$$a_{-1,s,\alpha}(q) = -ik_0 I_0 \pi a^2 [p_{s,\alpha}(q) - n_{s,\alpha}(q)] / N_{-1,\alpha}(q), \quad (22)$$

where use was made of the fact that $N_{-1,\alpha}(q) = N_{1,\alpha}(q)$. It is interesting that expressions (21) and (22) exactly coincide with those for an infinitesimal magnetic dipole having the moment $p_0^m = I_0 \pi a^2$ and rotating clockwise or counterclockwise, respectively, around the z axis. Such a dipole excites only one harmonic with the index $m = 1$ or $m = -1$.

Further simplifications are possible by allowing for the inequalities $|\varepsilon| \ll |g| \ll |\eta|$, which are valid in the frequency range (18). Using these inequalities, it is possible to show that the relation between p_e and q is well described by the following approximation [5]:

$$q^2 = g^2 / (p_e^2 - \varepsilon) - p_e^2 + \varepsilon. \quad (23)$$

Allowing for (21)–(23), one can obtain the power radiated from the source \mathbf{J}_- as

$$P \approx P_1 = |I_0|^2 Z_0 \frac{\pi^2 (k_0 a)^4}{32} \frac{g^2}{|\varepsilon|^{1/2}} \left(1 + \frac{22}{3\pi} \frac{|\varepsilon|^{1/2}}{|g|^{1/2}}\right). \quad (24)$$

For the source \mathbf{J}_+ , the radiated power takes the form

$$P \approx P_{-1} = |I_0|^2 Z_0 \frac{\pi^2 (k_0 a)^4}{32} \frac{g^2}{|\varepsilon|^{1/2}} \left(1 - \frac{26}{3\pi} \frac{|\varepsilon|^{1/2}}{|g|^{1/2}}\right). \quad (25)$$

It is seen from (24) and (25) that $P_1 \simeq P_{-1}$ and this relation remains valid even for larger sources with radius $a \lesssim (k_0 q_{\max})^{-1}$. This is not surprising if one takes into account that a relatively small magnetic-type source, regardless of the rotation direction of the equivalent magnetic dipole, excites most efficiently quasi-transversely

propagating whistler waves with $q \lesssim q_{\max}$, for which the magnetic-field ellipticity is fairly slight. However, for quasi-longitudinally propagating whistler waves, which are much poorly radiated from an electrically small antenna but have well-pronounced elliptical polarization, the excitation efficiency strongly depends on the source rotation direction.

These features can be elucidated in the difference of the respective field patterns near the z axis. As an example, Figs. 2 and 3 show the patterns of the magnetic-field magnitude (normalized to maximum value) of the sources \mathbf{J}_- and \mathbf{J}_+ , respectively, for parameters typical of the corresponding laboratory experiments (see, e.g., [1]): $\omega = 1.8 \times 10^6 \text{ s}^{-1}$, $\Omega_c = 1.2 \times 10^5 \text{ s}^{-1}$, $\omega_{\text{LH}} = 10^7 \text{ s}^{-1}$, $\omega_c = 8.8 \times 10^8 \text{ s}^{-1}$, $\omega_p = 1.6 \times 10^{10} \text{ s}^{-1}$, and $k_0 q_{\max} a = 0.95$. The field direction is depicted by the arrows in the figures.

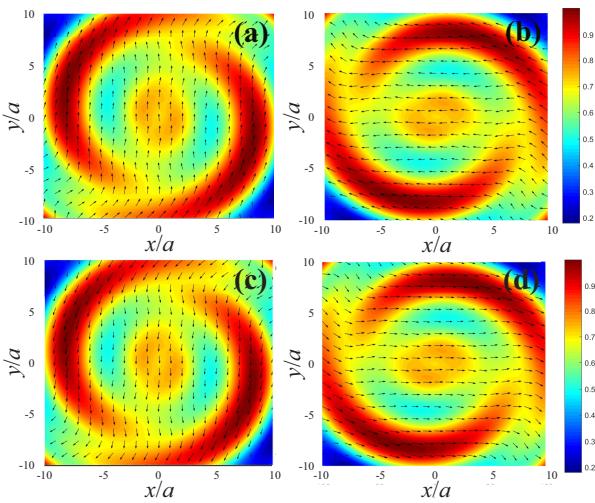


Figure 2. Magnetic field structure in the plane perpendicular to the z axis at the distance $z = 8\pi/k_0(\epsilon - g)^{1/2}$ from the source with the current density \mathbf{J}_- at time instants $t = t_0$ (a), $t = t_0 + T/4$ (b), $t = t_0 + T/2$ (c), and $t = t_0 + 3T/4$ (d), where $T = 2\pi/\omega$ and t_0 is the time at which the field \mathbf{H} on the z axis is aligned with the y_0 -direction.

It is evident from Fig. 2 that a well-pronounced helical spatial structure with the nonzero field on the z axis is typical of the source with clockwise rotation of the magnetic dipole, in contrast to the patterns in Fig. 3 for the opposite dipole rotation direction.

5 Conclusion

The radiation from the simplest nonsymmetric sources capable of exciting electromagnetic waves with helical phase fronts in a homogeneous magnetoplasma has been addressed. The approach developed is demonstrated to be convenient for full-wave analysis of the radiated power and field distributions of such sources and seems promising for application to more complicated systems including phased antenna arrays in a magnetoplasma.

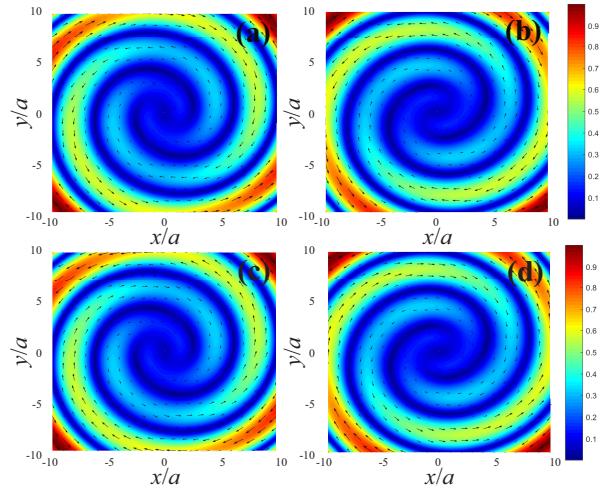


Figure 3. Same as in Fig. 2 but for a source with the current density \mathbf{J}_+ . The field behavior near the z axis is explained by that the counterclockwise rotating magnetic dipole does not radiate along this axis.

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