The method of mathematical modeling of wave fields and caustic structures in the process of propagation of electromagnetic radiation in the ionospheric plasma

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Abstract

The structure of the electromagnetic field arising at the Earth's surface during the propagation of decameter radio waves in the ionospheric plasma in the vicinity of caustics is studied and a new approach to the construction of uniform asymptotics based on symbolic calculations is developed. Comparison of the results of calculating the wave field in the caustic and ray approximation is performed. The amplitude structures of the field in the vicinity of the caustics of the o- and x- waves are considered, and the influence of the x-wave on the amplitude structure of the field of the o-wave is investigated. Based on the ray approximation, an estimate is proposed for the maximum value of the field amplitude in the vicinity of the caustic.

1 Introduction

With the propagation of electromagnetic waves in the decameter range, the problem arises of describing wave fields in focusing regions and, first, on caustics. In the vicinity of caustics, the amplitude of the wave field increases significantly, and caustics divide the physical space of the problem into regions with different propagation patterns [1-3].

From a mathematical point of view, caustics are envelopes of ray families, which can be constructed based on the solution of a bi-characteristic system. The bi-characteristic system describes the propagation of radio waves in the Earth's ionosphere, the initial conditions of which are determined by the radiation conditions in the source.

A new method of mathematical modeling of caustic structures (focusing regions) arising during the propagation of electromagnetic radiation in the ionospheric plasma in the presence of medium inhomogeneities has been developed, based on symbolic calculations. Calculation of wave fields on caustics is carried out to adequately interpret the ionograms of oblique and vertical sounding, considering the absorption and divergence of the radio signal in the ionospheric anisotropic plasma. A comparison is made of the wave field calculated in the ray approximation and in the caustic approximation. An estimate is obtained for the maximum value of the wave field amplitude in the vicinity of caustics for waves of different polarizations.

In the decameter range, the expression for the effective dielectric constant can be represented in the form of the Appleton-Hartley formula [4,5], which includes the function $N(\vec{r})$ – the electron concentration at a point in space with coordinates $\vec{r} = (x, y, z)$, and H_0 – the absolute value of the Earth's magnetic field, as well as the angle α between the wave vector and the vector $\vec{k}$ of the magnetic field strength. The orientation of the magnetic field strength is determined by the angles γ and φ.

When constructing ray trajectories, the method of the bi-characteristic Hamilton-Lukin system was used (see, for example, [6,7]) with initial conditions, which include the position of the radiation source and ζ, η – the initial angles of the ray exit. It is assumed that $\vec{r}$ – ray coordinates, t – group time, ω – angular frequency. The solution of the system is the functions (characteristics):

$$\vec{r} = \vec{r}(\zeta, \eta, t), \vec{k} = \vec{k}(\zeta, \eta, t) \quad (1)$$

To calculate the electromagnetic field in the vicinity of the caustic, a model of a mid-latitude daytime ionospheric plasma with a constant magnetic field was used. The height dependence of the electron concentration N is shown in Fig. 1. In Fig. 2 a graph of the effective frequency of collisions of electrons ν_e is shown (see [8,9]).

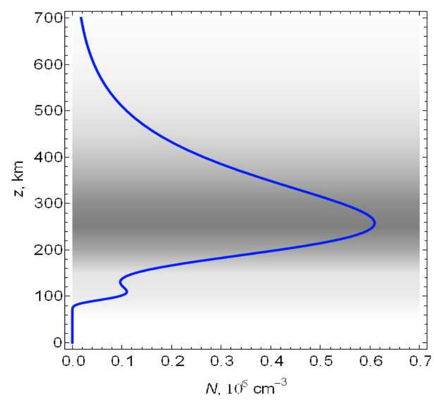

Figure 1. Graph of dependence of electron concentration on altitude.

In this work, a monochromatic radiation source was chosen, located on the earth's surface at the origin of coordinates with an operating frequency $f = 9.5$ MHz. The rays leave the source in the (x,z) plane, that is, $\eta = 0$. The figures show the distribution of the electron density in the ionosphere as a background.

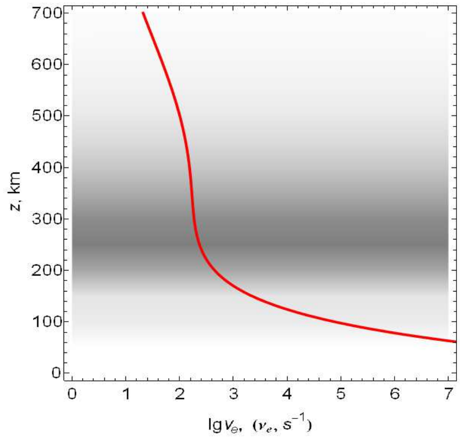

Figure 2. Graph of the effective frequency of collisions of electrons.

Figure 3 shows the ray paths in the plane (*x*,*z*) for an ordinary wave (o-wave). The angle ζ of the rays' exit varies from 0° to 90°. Ray families of x-waves (extraordinary waves) and o-waves form complex but similar caustic structures, each containing two caustic tips. Rays with large exit angles pass through the ionosphere, while those with small angles are reflected from the *E* or *F* layers and return to the ground.

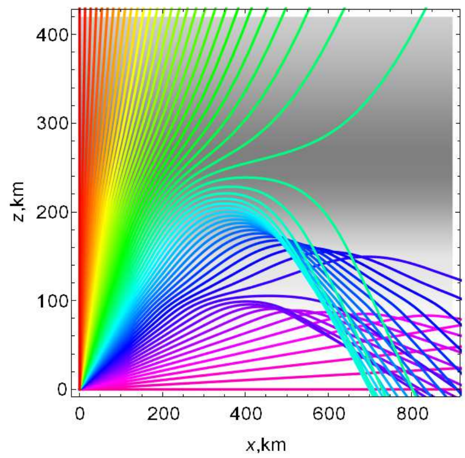

Figure 3. Ray paths in the plane (*x*,*z*).

The main maximum of the F-layer forms the upper caustic tip (cusp), and the E-layer forms the lower cusp. According to the classification of wave catastrophes, the cusp is the **A**$_3$ catastrophe [3,6].

The ray structure for an x-wave is similar to the ray structure of an o-wave, but the first caustic that crosses the Earth's surface is closer to the source in the case of an x-wave (642.5 km), and in the case of an o-wave it is farther (699.5 km) [9].

Let us consider the field amplitude in the vicinity of the caustic without taking into account the surface wave, the effect of which is negligibly small at the given distances from the radiation source at the considered frequency. When moving from the source along the surface, the caustic of an x-wave first appears, which is the boundary of the "dead zone", and then the caustic of an o-wave. The field of an x-wave is formed by two rays: a ray that has touched the caustic of the x-wave and a ray that has not touched this caustic. The caustic field of an o-wave interacts with the field of an x-wave. It is assumed that the radiation source is isotropic with a radiation power of 1 kW. Initial value of the field E_0.

2 Calculation methods

In the region of light (to the right of the extraordinary caustic), the geometrooptical (GO) field of the x-wave can be calculated as the sum of the contributions of two rays (κ= x):

$$u_g^\kappa \cong b_1^\kappa \cdot exp\big(i(\Phi_1^\kappa - \pi/2)\big) + b_2^\kappa \cdot exp(i\Phi_2^\kappa) \qquad (2)$$

The ray with index "1" in (2) touched the caustic, but the ray with index "2" did not. The GO field of an o-wave, which is formed to the right of the caustic of an o-wave, is also two-ray and is described by formula (2) at κ= o. The amplitude coefficients b_j are as follows:

$$b_j = E_0\, exp[-\psi_j]\sqrt{|J_0 J_j^{-1}|} \qquad (3)$$

In expression (3), the divergence Jacobian J_j and the initial value of this Jacobian J_0 were found based on the extended Lukin's bi-characteristic system of [7,8], which additionally contains 12 more equations with the corresponding initial conditions. The absorption ψ_j, determined by the electron collision frequency, and the phase Φ_j are calculated along the ray path [5, 8, 9]. Since the Jacobian J_j is zero on the caustic, the coefficients b_j become infinite and the GO solution loses its meaning. Therefore, the field in the vicinity of the caustic should be found using the uniform asymptotics (see [6,10]):

$$u_c \cong exp(i\theta)(l_1 \cdot Ai(\lambda) - il_2 \cdot dAi(\lambda)/d\lambda) \qquad (4)$$

In formula (4) θ is the traveling wave phase,

$$Ai(\lambda) = \int_{-\infty}^{+\infty} exp(i(\xi^3 + \lambda\xi))\, d\xi \qquad (5)$$

is the Airy function and λ is Airy function argument, At $\lambda < 0$ (in the region of light) we find:

$$\theta = 0.5(\Phi_1 + \Phi_2),\ \lambda = -3 \cdot 2^{-4/3}|\Phi_1 - \Phi_2|^{2/3} \qquad (6)$$

The coefficients l_1 and l_2 in the first approximation are equal:

$$l_1 \cong \frac{b_1+b_2}{2\sqrt{\pi}}\sqrt[4]{-3\lambda}, \quad l_2 \cong \frac{3(b_1-b_2)}{2\sqrt{\pi}}\frac{1}{\sqrt[4]{-3\lambda}} \qquad (7)$$

Since the phase of the traveling wave, the coefficients of the asymptotic expansion and the argument of the Airy function are determined through the amplitudes and phases of two rays arriving at the same point, it becomes necessary to calculate at each point the amplitudes and phases of these rays arriving along close but different trajectories with very high accuracy. This approach is known as the " shooting" problem.

In this paper, a hybrid algorithm is implemented that contains both elements of the “saddle point” method and the method of local asymptotics [11] and is close to the interpolation approach proposed in [13]. First, the coordinate of the caustic x_c on the x-axis is determined and the angle ζ_c of exit of the ray touching this caustic in x_c is found. Two subfamilies of rays arise: rays for which the angle $\zeta < \zeta_c$ (touched by caustics), and rays for which $\zeta > \zeta_c$ (not touched caustics). After determining the point of ray intersection with the earth's surface, the parameters were calculated: time of arrival, phase, absorption, amplitude, etc. Then, interpolation formulas were found for the phases and amplitude coefficients for each ray subfamily. Based on these formulas, the parameters of two intersecting rays were calculated and the ray and caustic fields were determined.

3 Simulation results

The amplitude of the x-wave in the vicinity of the caustic is shown in Fig. 4. The caustic solution calculated using the Airy function and its derivative is shown by the solid line, and the GO solution is shown by the dotted line. Absorption was not taken into account in the calculations for the upper curves, and it was taken into account for the lower ones. It follows from the calculations that the maximum of the amplitude is shifted relative to the position of the caustic, marked by a dot, into the region of light [9].

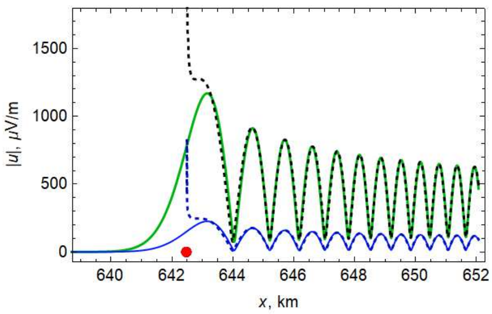

Figure 4. Amplitude of the field of an x-wave in the vicinity of the caustic.

The field amplitude values at the caustic are close to the average amplitudes in the light region. The maximum field value for the curves with absorption c is higher in the case of an o-wave. The amplitude oscillations are ~ 1 km and decrease with distance from the source. Comparison of the field amplitude at the caustic, calculated using the uniform (caustic) asymptotic formulas (4), and the GO approximation (2) (dashed line) in the light region showed very good agreement up to the slope of the main maximum of the field amplitude (Fig. 4). Therefore, it is sufficient to determine the inflection point of the curve before it goes to infinity and use this value to estimate the maximum value of the field amplitude in the vicinity of the caustic using the GO approximation. For an o-wave, calculations give similar results.

The amplitudes of the x-wave (blue) and o-wave (green) over the entire investigated range along the earth's surface are shown in Fig. 5. Absorption taken into account. It follows from the calculations that when absorption is not taken into account, the amplitudes of the fields of the x- and o-waves in the vicinity of the caustic do not differ greatly.

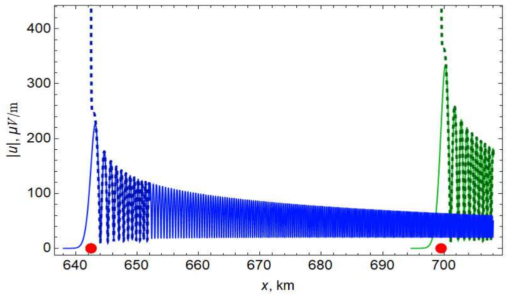

Figure 5. Amplitudes of the field of x- and o-waves with absorption.

When absorption is taken into account, the field amplitude in the vicinity of the caustic is much larger for an o-wave than for an x-wave. The period of oscillations decreases with distance from the caustic. The maximum values of the field amplitude of the x-wave decrease significantly, while the minimum values increase. Therefore, the range of oscillations decreases. The amplitude of the total field of the x- and o-waves with absorption in the vicinity of the caustic of the o-wave without considering the polarization is shown in Fig. 6.

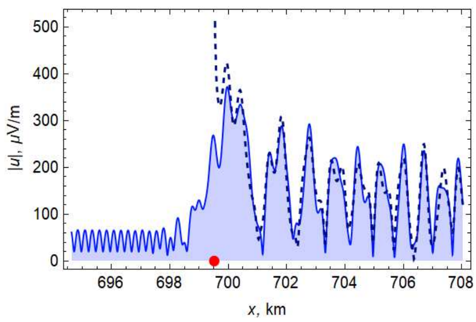

Figure 6. The amplitude of the field of the x- and o-waves with absorption near the caustic.

The figure allows one to estimate the influence of an x-wave in the vicinity of the caustic of an o-wave while maintaining coherence. At $x < 698$ km, regular oscillations are visible, formed by a two-ray field of an o-wave. If absorption is not accounted for, the major amplitude is heavily indented and comparable to other amplitudes. The last minimum of the GO field is used to judge the maximum real value of the field amplitude near the caustic in the case of absorption when the classical structure of the field amplitude in the vicinity of the caustic is well preserved [9].

The amplitude of the modulus of the total field strength of the x- and o-waves with absorption (lower blue line with shading) and without absorption (green line) are compared in Fig. 7.

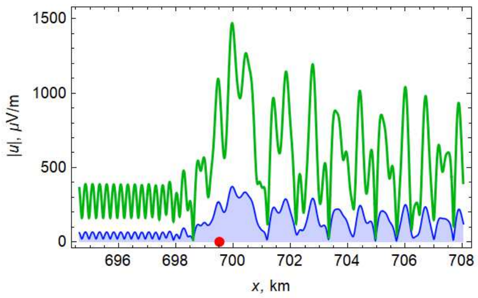

Figure 7. Amplitudes of the field of the x- and o-waves with and without absorption near the caustic.

The structure of the field amplitude without taking into account absorption and taking into account absorption differ by more than three times, but they are qualitatively the same. Significant differences are due to the different influence of absorption on the x- and o-waves [9].

4 Conclusion

Thus, the structure of the field of an electromagnetic wave propagating in the ionospheric plasma in the vicinity of the caustic has been investigated taking into account the external magnetic field. The comparison of the caustic structure of the field of the o- and x-waves is carried out, the influence of the x-wave on the amplitude field of the o-wave in the caustic vicinity is estimated. The wave fields calculated by the formulas of the uniform (caustic) approximation and the nonuniform (ray, GO) approximation are compared. It is shown how the maximum value of the field amplitude in the vicinity of the caustic can be estimated from the GO approximation, without resorting to uniform asymptotics, from the position of the inflection point of the graph of the amplitude of the ray solution.

5 Acknowledgements

This work was supported by the Russian Science Foundation (grant no. 20-12-00299).

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