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An Application of Monte Carlo Markov Chains in Inverse Electromagnetic Scattering Problems

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Abstract

To tackle the nonlinearity in inverse electromagnetic scattering problems, we present a novel application of Markov Chain Monte Carlo (MCMC) methods to infer permittivity values. Instead of using regularizations to uncover the best fit of permittivity, we estimate the conditional mean of the unknown permittivity given scattered field data. The conditional mean estimates not only incorporate prior knowledge from results obtained by Born Iterative Methods (BIM), but also avoid the nonlinearity by computing the linear forward model. For a homogeneous cylinder with the relative permittivity of 11, numerical results of BIM are improved by MCMC.

1 Introduction

Inverse electromagnetic scattering problems arise in many scientific and engineering fields [1, 2, 3]. Given a scatterer immersed in free space and an incident field, the electromagnetic scattering phenomenon is characterized by the electric field integral equation as

$$\mathbf{E}^s(\mathbf{r}) = -k_0^2 \int_V (\varepsilon(\mathbf{r}')-1)\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}(\mathbf{r}')\,\mathrm{d}V',\ \mathbf{r}\notin V, \quad (1)$$

where $\mathbf{E}^s(\mathbf{r})$ is the scattered electric field, k_0 is the wave number in free space, $\varepsilon(\mathbf{r}')$ is the complex permittivity profile of the scatterer, $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ is the Green's function, and $\mathbf{E}(\mathbf{r}')$ is the total field within the scatterer. The time harmonic $\exp(j\omega t)$ is omitted throughout this paper.

For a forward problem, one aims to compute the scattered field with a known permittivity profile. As $\mathbf{E}(\mathbf{r}')$ is numerically determined from the known $\varepsilon(\mathbf{r}')$, $\mathbf{E}^s(\mathbf{r})$ can be solved with a fixed integrand. For an inverse problem, given the scattered electric field, we aim to reconstruct the complex permittivity profile. As $\mathbf{E}(\mathbf{r}')$ becomes an unknown due to the unknown $\varepsilon(\mathbf{r}')$, the inverse problem for the permittivity is nonlinear.

Some previous deterministic approaches linearize the nonlinear inverse scattering problem. For example, the first-order Born approximation corresponds to the initial step of Born-related methods such as the Born iterative method (BIM) [4] and the distorted Born iterative method [5]. Similarly, higher-order approximations do not linearize our problem, yet provide more accurate representations of unknown fields [6], which result in better reconstructions. Some previous stochastic approaches combine Born approximation and natural computing techniques, such as genetic algorithms [7] and memetic algorithms [8]. Also Bayesian inference has been widely used in parameter estimation, for example, in electrical impedance tomography [9].

In this work, we extend the work presented in [10]. First, we adapt BIM results as priors by treating the unknown permittivity of a two-dimensional scatterer and known scattered field data as random variables. Then we sample the conditional mean (CM) integration with a simple MCMC method, using the Metropolis-Hastings algorithm to construct the chain by random walks with Gaussian proposal functions. Numerical computations based on the Method of Moment (MoM) for a homogeneous lossless scatterer with a known contour are performed.

2 Theory

To numerically solve two dimensional inverse scattering problems, one discretizes Eq. 1 into matrix form with the Method of Moment [11]. Let there be N_{tx} transmitters and N_{rx} receivers around the scatterer under investigation; there are $N_{tx}\times N_{rx}$ measurements. Given the tomographic configuration, the scatterer is illuminated by one transmitter each time, and the corresponding scattered fields are measured by all receivers. Therefore, multi-illumination and multi-static measurements are achieved to infer information about this scatterer. Then the domain of the scatterer is discretized into n pixels. At the m-th measurement, the scattered field E^s can be rewritten as

$$E_m^s = -\frac{jk_0^2}{4}\sum_{i=1}^{n}(\varepsilon_i-1)E_i\int_{S_i}\mathrm{H}_0^{(2)}(k_0|\mathbf{r}_m-\mathbf{r}_i'|)\,\mathrm{d}S', \quad (2)$$

where $\mathrm{H}_0^{(2)}$ is the Hankel function of the second kind, $\mathbf{r}_m\notin S$. By enforcing the "measurement" location to be within the scatterer, the total field at the p-th pixel, E_p, can be represented as

$$E_p = E_p^i - \frac{jk_0^2}{4}\sum_{i=1}^{n}(\varepsilon_i-1)E_i\int_{S_i}\mathrm{H}_0^{(2)}(k_0|\mathbf{r}_p-\mathbf{r}_i'|)\,\mathrm{d}S', \quad (3)$$

where E_p^i is the incident field at p-th pixel, $\mathbf{r}_p\in S$.

The BIM routine starts with the Born approximation in Eq. 2 to compute the permittivity profile, which then updates the field within the scatterer by Eq. 3. In this work, we use the conjugate gradient (CG) method as the regularization technique to solve the inversion in Eq. 2 due to the ill-conditioned property of the linear operator. After the BIM method provides stable outputs, we reformulate this inverse problem from a Bayesian inference perspective and consider measurements and parameters of a statistical model as random variables.

2.1 Bayesian Inference and MCMC

The Bayes theorem states that the distribution of unknown permittivity parameters $\mathbf{X}$ conditioned on scattered field data $\mathbf{D}$ is

$$P(\mathbf{X}|\mathbf{D}) = \frac{P(\mathbf{X})P(\mathbf{D}|\mathbf{X})}{P(\mathbf{D})}. \tag{4}$$

In Bayesian inverse models, the solution of an inverse problem takes the form of a posterior probability distribution, $P(\mathbf{X}|\mathbf{D})$, which is proportional to the prior multiplied by the likelihood. The likelihood, $P(\mathbf{D}|\mathbf{X})$, of data $\mathbf{D}$ given permittivity parameters $\mathbf{X}$ is strongly associated with the forward scattering model. Plausible priors of $\mathbf{X}$ are: 1) the real part of the relative permittivity being more than or equal to 1 and the imaginary part being negative; 2) previous reconstruction results from deterministic conjugate gradient regularizations offering upper and lower bounds.

To estimate the permittivity parameters, we choose the conditional mean, i.e. the center of the posterior probability distribution, of the unknown model parameter $\mathbf{X}$,

$$\mathbf{X}_{CM} = E\{\mathbf{X}|\mathbf{D}\} = \int_{\mathbb{R}^n} \mathbf{X}\pi(\mathbf{X}|\mathbf{D})\,\mathrm{d}\mathbf{X}, \tag{5}$$

where $\pi(\mathbf{X}|\mathbf{D})$ is the posterior density. The conditional mean estimate solves an integration problem, so usually it is more robust towards noises in the data than the maximum a posteriori estimate.

Since the high dimension of the unknown discrete parameters requires a large sample space, it is challenging to numerically evaluate the integration for a conditional mean estimate. Here, the conditional mean is sampled in a statistical sense using MCMC methods, which can be applied favorably for our nonlinear inverse problem as they only depend on the forward model. The Monte Carlo integration draws samples from the posterior probability density and takes the average of these samples. Thus, we approximate the integral in Eq. 5 with the population mean

$$E\{\mathbf{X}|\mathbf{D}\} \approx \frac{1}{N}\sum_{t=1}^{T} \mathbf{X}_t\pi(\mathbf{X}_t|\mathbf{D}). \tag{6}$$

To sufficiently draw samples from posterior distributions, we construct a Markov chain. With the Metropolis-Hastings (MH) algorithm, a Markov chain converges to its stationary distribution, which is also the posterior distribution we are trying to sample. At each time t, one samples $\mathbf{Y}$ from the proposal distribution. According to the acceptance ratio, the sample $\mathbf{Y}$ is either accepted as the next state $\mathbf{X}_{t+1}$ or not. In this work, we use the simplest MH algorithm, the random walk with a multivariate Gaussian proposal function, which has the proposal distribution,

$$q(\mathbf{Y}|\mathbf{X}) = q(|\mathbf{Y}-\mathbf{X}|). \tag{7}$$

This is a trial-and-error strategy; at each state t, we add some randomness to $\mathbf{X}_t$ so that the proposed sample $\mathbf{Y}$ explores the solution space.

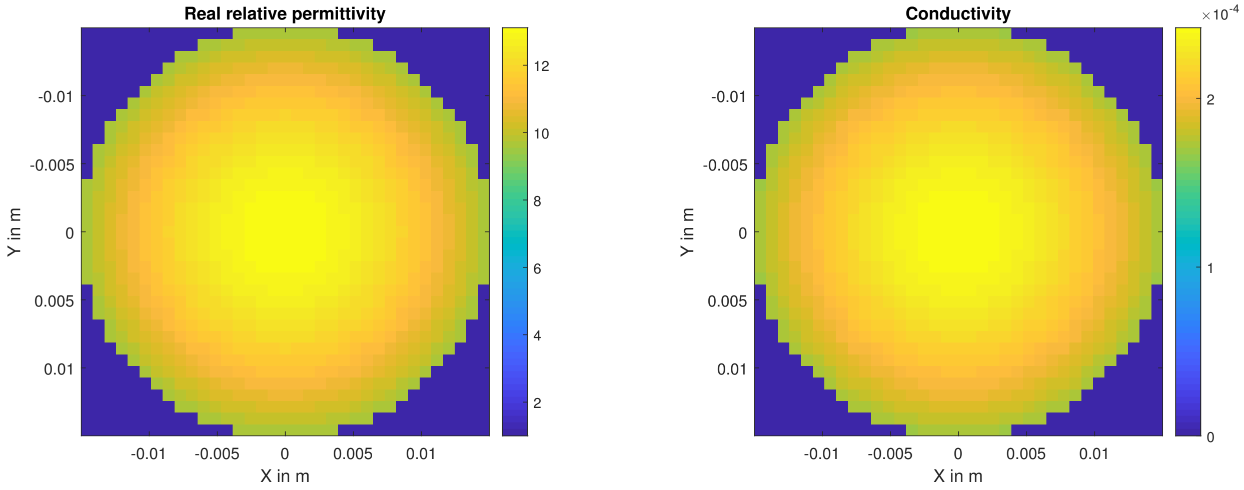
3 Numerical Results and Analysis

For a simple test case, we choose an infinitely long circular cylinder with the radius of $\lambda/20$ and the relative permittivity of 11. This object is borrowed from [4]. A square that contains the circle with the side length of 0.03 m is the investigation domain. To avoid the inverse crime, the investigation domain, S, is divided into finer pixels (144×144) in the forward model than the inverse one (36×36). There are 8 transmitters and 36 receivers that offer 288 measurements. TM-mode incident fields are radiated by a line source at 1 GHz and scattered fields are numerically calculated by MoM codes.

First, given the contour of the scatterer, we perform the traditional BIM with the conjugate gradient; Figure 1 shows the BIM results after 11 iterations: a rough range for the real part of the permittivity and quite accurate reconstructions of the conductivity.

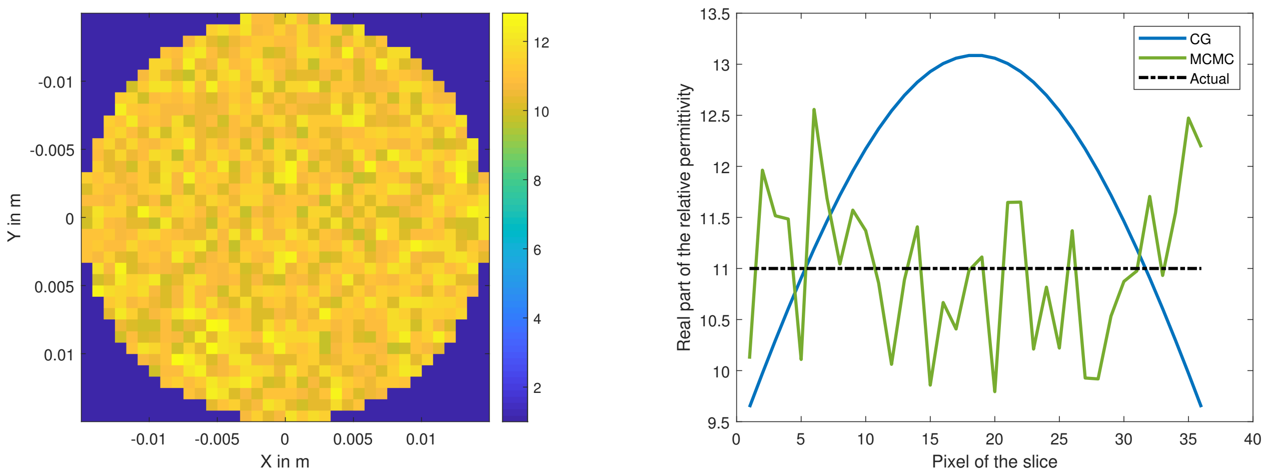
Then, for the real part of permittivity, we set the max/min values acquired in BIM as the upper/lower bounds for the permittivity random variable; for the imaginary part, we assume it is 0. The starting point of the Markov chain is the mean of the real permittivity at all pixels in the scatterer. As previous BIM results offer a good starting point, we don't throw away any iterations at the beginning of the chain; therefore no burn-in phase is need. At each iteration, a random variable of the normal distribution with a standard deviation of 0.02 is added to the current state of permittivity values; this procedure generates a sample permittivity, which would be sent to the forward model to obtain the sample data. The difference between these sample data and scattered field data determines if this sample permittivity would be accepted as the next state of the chain. The longer the chain is, the closer the estimate is to the true posterior.

Figure 2 shows the reconstructed results of MCMC after 10000 iterations. The acceptance rate is 0.233. Due to the random nature of MCMC, the reconstructed permittivity profile in Figure 2(a) is not as smooth as the conjugate gradient results in Figure 1(a). However, even if the lower/upper bounds are set as [9.5, 13], the conditional mean estimate by MCMC offers narrower bounds as [9.766,12.815]. Moreover, the mean of the permittivity at all pixels by BIM is 11.180, which is improved by MCMC as 11.022; the standard deviation by BIM is 1.043, which is also improved by MCMC as 0.669. Figure 2(b) directly



(a) **(b)**

Figure 1. BIM reconstructed results for the permittivity within a cylinder, $\varepsilon = 11$: (a) real part; (b) conductivity.



(a) **(b)**

Figure 2. Reconstructed results for real part of the permittivity: (a) MCMC results within the cylinder; (b) slice comparisons along the horizontal axis.

compares the permittivity reconstructions along 36 horizontal pixels at $y = 0$ by BIM with the conjugate gradient and MCMC.

Furthermore, we insert those reconstructed permittivity values into the forward model to compute scattered fields, which are compared with the analytical scattered field shown as in Figure 3. Not surprisingly, for forward model results, MCMC improves both of the amplitude and the phase of scattered fields. Since scattered fields are complex, we compare the amplitude by calculating the error as

$$\text{error} = \left| \frac{E^s_{\text{CG,MCMC}} - E^s_{\text{data}}}{E^s_{\text{data}}} \right|. \quad (8)$$

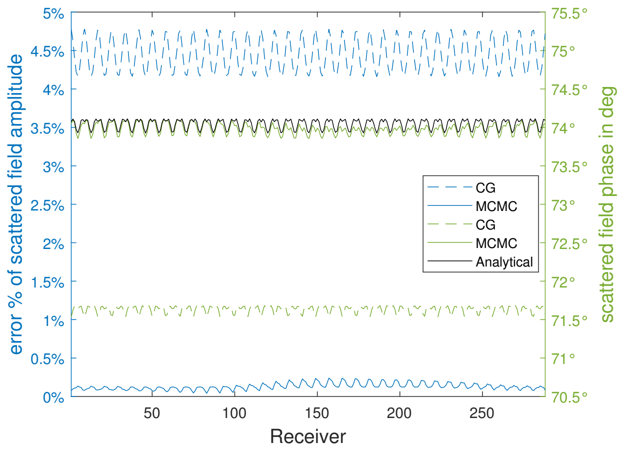


Figure 3. Comparison of scattered fields from the forward model. Left axis: error percentage of amplitude. Right axis: phase.

4 Conclusions

Stochastic Bayesian inference is applied with priors from the BIM with the conjugate gradient technique. MCMC is computationally expensive compared to deterministic regularizations; however, MCMC does improve the reconstructed permittivity profile for a scatterer with $\varepsilon = 11$, where the Born approximation might fail. Future work would include applying MCMC to inhomogeneous scatterers and adding noise to data for robustness tests.

References

[1] M. Pastorino, *Microwave Imaging*, Wiley, March 2010, doi: 10.1002/9780470602492.

[2] L. Lo Monte, D. Erricolo, F. Soldovieri and M. C. Wicks, “Radio Frequency Tomography for Tunnel Detection," *IEEE Transactions on Geoscience and Remote Sensing*, **48**, 3, March 2010, pp. 1128–1137, doi: 10.1109/TGRS.2009.2029341.

[3] L. Lo Monte, F. Soldovieri, D. Erricolo and M. C. Wicks, “Imaging Below Irregular Terrain Using RF Tomography," *IEEE Transactions on Geoscience and Remote Sensing*, **50**, 9, Sept. 2012, pp. 3364–3373, doi: 10.1109/TGRS.2012.2183136.

[4] Y. M. Wang, and W. C. Chew, “An iterative solution of the two-dimensional electromagnetic inverse scattering problem,” *Int. J. Imaging Syst. Technol.*, **1**, 1989, pp.100–108, doi:10.1002/ima.1850010111.

[5] W. C. Chew and Y. M. Wang, “Reconstruction of two-dimensional permittivity distribution using the distorted Born iterative method," *IEEE Transactions on Medical Imaging*, **9**, 2, June 1990, pp. 218–225, doi: 10.1109/42.56334.

[6] V. Picco, G. Gennarelli, T. Negishi, F. Soldovieri and D. Erricolo, “Experimental Validation of the Quadratic Forward Model for RF Tomography," *IEEE Geoscience and Remote Sensing Letters*, **12**, 7, July 2015, pp. 1461–1465, doi: 10.1109/LGRS.2015.2409025.

[7] S. Caorsi and M. Pastorino, “Two-dimensional microwave imaging approach based on a genetic algorithm," *IEEE Transactions on Antennas and Propagation*, **48**, 3, March 2000, pp. 370–373, doi: 10.1109/8.841897.

[8] S. Caorsi, A. Massa, M. Pastorino, M. Raffetto and A. Randazzo, “Detection of buried inhomogeneous elliptic cylinders by a memetic algorithm," *IEEE Transactions on Antennas and Propagation*, **51**, 10, Oct. 2003, pp. 2878–2884, doi: 10.1109/TAP.2003.817984.

[9] J. P. Kaipio, V. Kolehmainen, E. Somersalo, and M. Vauhkonen, “Statistical inversion and Monte Carlo sampling methods in electrical impedance tomography”, *Inverse problems*, **16**, 5, 2000, pp. 1487-1522.

[10] Y. Liu, D. Erricolo, “Two-dimensional inverse scattering using a Bayesian approach,” *2019 URSI Commission B International Symposium on Electromagnetic Theory (EMTS 2019)*,” San Diego, CA, May 27-31, 2019.

[11] J. Richmond, “Scattering by a dielectric cylinder of arbitrary cross section shape,” *IEEE Transactions on Antennas and Propagation*, **13**, 3, May 1965, pp. 334–341, doi: 10.1109/TAP.1966.1138730.