## A Simplified Model of Microwave Coupling in ECR Plasmas

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#### Abstract

A magnetized plasma strongly interacts with microwaves, when their angular frequency $\omega$ is around the electron cyclotron resonance (ECR) one, that is $\Omega=e\left|\mathbf{B}^{s}\right| / m_{e}$, with $\mathbf{B}^{s}$ the static flux density. The case of small ion sources with $\mathbf{B}^{s} \cong 0.5$ to 1 T , where wavelength and plasma chamber radius are comparable, is studied with full wave equations, assuming a collision rate $v$ and a plasma of given density. Wave initially launched in the TE mode from a waveguide ( $\mathrm{TE}_{11}$ for a circular waveguide), aligned with the axis $z$ of a cylindrical plasma chamber and of $\mathbf{B}^{s}$, are partially converted to $E_{z}$ waves; use of azimuthal symmetry to consider only the $m=1$ component is discussed. Large reflections are sometimes observed as a function of ratio of plasma angular frequency $\omega_{p}(z)$ and $\omega$, possibly due to model simplification (density constant in $x$ and $y$ ). Extension to off-axis waveguides (perhaps rectangular or coaxial) is mentioned.


## 1 Introduction

ECR (Electron Cyclotron Resonance) ion sources[1] are standard suppliers of multiply charged ions for accelerators: plasma electrons are efficiently heated by microwave while spiraling in a static magnetic field $\mathbf{B}^{s} / \mu_{0}$ which also provides for their confinement. Same resonance is used for tokamak heating. While plasma chamber dimension (radius $R_{c}$ or minor radius) is much larger than wavelength $\lambda$ in latter case (where quasi-optical microwave study is possible) and for large ion sources, for compact ion sources [2, 3, 4] values of $R_{c}$ and $\lambda$ may be comparable, so that full wave equations must be solved. Some schematic geometries are shown in Fig. 1; in our case the plasma chamber is a cylinder with axis $z$, length $L_{p}=0.18 \mathrm{~m}$ and $R_{c}=0.03 \mathrm{~m}$. The input waveguide may be a WR62, typically off-axis[2], or a coaxial line[1]; in this preliminary study we simply consider a radius $b$ circular waveguide centered on axis $z$; in most examples $b=9.25 \mathrm{~mm}$ (so fundamental mode has the same cutoff frequency as in a WR62) and $B^{s}>0.45$ T. Waveguide input is at the $z=-L_{g}$ plane, with $L_{g}=0.06$ m , while its junction with plasma chamber is in the $z=0$ plane.

## 2 Basic equations and assumptions

We consider that an assigned static magnetic field $\mathbf{B}_{s} / \mu_{0}$ is applied to plasma, plus an unknown radiofrequency with


Figure 1. (a) $r z$ section of plasma chamber and circular waveguide; (b) its $z=0$ section, with metal walls hatched; (c) as 'b' but with a rectangular waveguide off-axis; (d) as 'b' but with a coaxial waveguide.
angular frequency $\omega$, magnetic induction $\mathbf{B}$ and electric field $\mathbf{E}$, much weaker than the static field, that is $\mathbf{E} \cong$ $O(c \mathbf{B}) \ll c\left|\mathbf{B}^{s}\right|$ (larger than $1.3 \times 10^{8} \mathrm{~V} / \mathrm{m}$ ). At GHz frequencies [in our example $\omega /(2 \pi)=14.4 \mathrm{GHz}$ ], plasma ions are considered at rest, while electrons move as free charges in vacuum giving a current density $\mathbf{j}=\sigma * \mathbf{E}$ where conductivity $\sigma$ in general is an integral operator and terms as $E^{2}$ are neglected since $\mathbf{E} \ll c \mathbf{B}^{s}$. The simplification of $\mathbf{j}$ locally dependent on $\mathbf{E}$ is often used[5], so $\mathbf{j}=\sigma \cdot \mathbf{E}$ with $\sigma$ a matrix; the charge density is then $\rho=\mathrm{i} \operatorname{div} \mathbf{j} / \omega$ understanding a time dependence $\exp (i \omega t)$ (as in phasor notation[6]).

There are two equivalent approaches to account for $\mathbf{j}$; one is to consider vacuum fields $\mathbf{H}=\mathbf{B} / \mu_{0}$ and $\mathbf{D}=\varepsilon_{0} \mathbf{E}$ and $\mathbf{j}$ as a free current; this gives

$$
\begin{equation*}
\operatorname{curl} \mathbf{H}=\mathbf{j}+\mathrm{i} \omega \mathbf{D}=\mathrm{i} \omega \varepsilon_{0} K \cdot \mathbf{E} \quad, \quad K=I_{3}-\mathrm{i} \sigma /\left(\omega \varepsilon_{0}\right) \tag{1}
\end{equation*}
$$

with $I_{3}$ the identity matrix, which introduces the effective dielectric tensor $K$. The other approach considers a plasma as a medium with effective electric displacement $\check{\mathbf{D}}=\varepsilon_{0} K \cdot \mathbf{E}$, no other free current $\check{\mathbf{j}}=0$ or charge $\check{\rho}=0$; this is manifest in the second step of eq. (1), which so represents both approaches. By substitution of $\operatorname{curl} \mathbf{E}=-i \omega \mathbf{H} / \mu_{0}$ in eq. (1), it follows the wave equation

$$
\begin{equation*}
\text { curl } \operatorname{curl} \mathbf{E}=(\omega / c)^{2} K \cdot \mathbf{E} \tag{2}
\end{equation*}
$$

from it or eq. (1) we get the constraint $\operatorname{div} K \cdot \mathbf{E}=0$, very helpful in numerical analysis and to supplement boundary
conditions. Moreover, in an empty waveguide, electric vector potential $\mathbf{F}$ is commonly defined and used[6]; for plasma filled waveguides, this constraint shows that $K \cdot \mathbf{E}$ is curl of some vector field; we thus generalize electric vector potential so that

$$
\begin{equation*}
\mathbf{E}=-K^{-1} \cdot \operatorname{curl} \mathbf{F} \tag{3}
\end{equation*}
$$

where $\mathbf{F}$ has manifestly dimension of [V]; but due to complicate expressions of $K^{-1}$ (and its derivatives) use of $\mathbf{F}$ in construction of solutions appears challenging.

For a cold magnetized plasma[5, 7] $K$ can be obtained as follows: to reach stationary plasma conditions, electron energy should have some dissipation mechanism, here for simplicity a friction force $F_{f}=-m_{e} v \mathbf{v}$, where the collision frequency $v$ is taken as a constant independent of velocity $\mathbf{v}$ (as in electron-neutral collision average), for a preliminary sensitivity study on $v$. Current density is then $\mathbf{j}=-n_{e} e \mathbf{v}$ with $n_{e}$ the electron density and motion equation gives

$$
\begin{equation*}
(\mathrm{i} \omega+v) \mathbf{v}=-\left(e / m_{e}\right)\left(\mathbf{E}+\mathbf{v} \times \mathbf{B}^{s}\right) \tag{4}
\end{equation*}
$$

This introduces the complex frequency $w=\omega-\mathrm{i} v$ and two real positive angular frequencies:

$$
\begin{equation*}
\omega_{p}=\left[\frac{n_{e} e^{2}}{\varepsilon_{0} m_{e}}\right]^{1 / 2} \quad, \quad \Omega_{c}=\frac{e\left|\mathbf{B}^{s}\right|}{m_{e}} \tag{5}
\end{equation*}
$$

where $\Omega_{c}$ is the cyclotron frequency and $\omega_{p}$ the plasma frequency. Solving eq. (4) for $\mathbf{v}$ and calculating $\mathbf{j}$ gives $K$ and its inverse $M=K^{-1}$ as
$K=\left[\begin{array}{ccc}k_{1} & k_{2} & 0 \\ -k_{2} & k_{1} & 0 \\ 0 & 0 & k_{3}\end{array}\right] \quad, \quad M=\left[\begin{array}{ccc}k_{1} / k_{s} & -k_{2} / k_{s} & 0 \\ k_{2} / k_{s} & k_{1} / k_{s} & 0 \\ 0 & 0 & 1 / k_{3}\end{array}\right]$
where the coordinate 3 (that is $z$ ) is parallel to $\mathbf{B}^{s}$ and 1 and 2 are orthogonal; here $k_{3}=1-\omega_{p}^{2} /(w \omega)$ and $k_{s}=\left(k_{1}^{2}+k_{2}^{2}\right)$, with

$$
\begin{equation*}
k_{1}=1-\frac{\omega_{p}^{2}}{\omega} \frac{w}{w^{2}-\Omega_{c}^{2}} \quad, \quad k_{2}=-\frac{\omega_{p}^{2}}{\omega} \frac{\mathrm{i} \Omega_{c}}{w^{2}-\Omega_{c}^{2}} \tag{7}
\end{equation*}
$$

where the Appleton ionospheric parameters $X_{A}=\omega_{p}^{2} /(\omega w)$ and $Y_{A}=\Omega_{c} / w$ are clearly recognisable.

## 3 Boundary conditions (bc) and solution

To reliably solve eq. (2) we need to set boundary conditions (bc) for its 3 three components (say $E_{r}, E_{\theta}$ and $E_{z}$ ), in the Neumann or Dirichlet form. On metal walls (of infinite conductivity) we have the tangential component $\mathbf{E}_{\|}=0$ which provides only two conditions; note they imply $B_{n}=0$, where ${ }_{n}$ indicates the normal component (by convention directed outwards [8]). A third condition comes from $\operatorname{div} K \cdot \mathbf{E}=0$. Just to simplify algebra, we assume that $\mathbf{B}^{s}$ is always directed along $z$, which is approximately true for the ECRIS solenoid; most ECRIS have also a sextupole field which is here neglected; moreover, we consider that $B_{z}$ and $\omega_{p}$ are not functions of $x$ and $y$. Then, on the $z=0$


Figure 2. Profiles $p_{b}=\Omega_{c}(z) / \omega$ and $p_{p}=\omega_{p} / \omega$ vs $z$.
or $z=L_{p}$ faces (set bc1), we have that fields $E_{\vartheta}=0$ and $E_{r}=0$ vanish, as well as their derivatives respect to $r$ and $\vartheta$ (written as $E_{r, r}$ and $E_{r, \vartheta}$ and similarly); then

$$
\begin{equation*}
0=\operatorname{div} K \cdot \mathbf{E}=\left(k_{3} E_{z}\right)_{, z} \rightarrow E_{z, z}=-E_{z}\left(k_{3, z} / k_{3}\right) \tag{8}
\end{equation*}
$$

which counts as a Neumann mixed[8] condition for $E_{z}$. On the $r=R_{c}$ or $r=b$ surfaces (set bc2) we have $E_{z}=E_{\vartheta}=0$ and third condition reduces to

$$
\begin{equation*}
0=k_{1}\left(r E_{r}\right)_{, r}+k_{2}\left(r E_{\vartheta, r}-E_{r, \vartheta}\right) \tag{9}
\end{equation*}
$$

which (even if complicate) still gives a Neumann condition for $E_{r}$ provided that $k_{1} \neq 0$ (generally true). These conditions greatly simplify for an empty waveguide since $k_{1}=1=k_{3}$ and $k_{2}=k_{3, z}=0$ hold there. Figure 2 shows typical profiles of $p_{b}(z)=\Omega_{c}(z) / \omega$ and $p_{p}(z)=\omega_{p}(z) / \omega$; note that $B_{z} \propto p_{b}$ and $n_{e} \propto p_{p}^{2}$; the latter is parametrized by its maximum $o_{l}$ at $z=L_{p} / 2$, its end value $o_{h}$ and its maximum inside waveguide $o_{w}$; for $z \geq 0$ we set $p_{p}=$ $o_{l}+\left(o_{h}-o_{l}\right)\left(z-L_{h}\right)^{2} / L_{h}^{2}$ with $L_{h}=L_{p} / 2$, while for $z<0$ we have

$$
\begin{equation*}
p_{p}(z)=\left(1+\left(z / L_{g}\right)\right)^{2} \Theta_{s}\left(z+L_{g} / 2, L_{t}\right) \tag{10}
\end{equation*}
$$

where $\Theta_{s}$ is the Heaviside function, smoothed within a length $L_{t}=0.01 \mathrm{~m}[8]$. The first factor gives a quadratic rise and the second makes $p_{p}=0$ for $z \leq-\frac{1}{2} L_{g}-L_{t}$, where waveguide is left empty and we can launch a $\mathrm{TE}_{11}$ wave; shown profile has $o_{l}=2$ (very large value, changed during scans) and $o_{w}=o_{h}=0.4 o_{l}$. The profile for $B_{z}$ is kept simpler and fixed, with $B_{z}=B_{h}=1.04 \mathrm{~T}$ for $z<0$ and $B_{z}=B_{o}+\left(B_{h}-B_{o}\right)\left(z-L_{h}\right)^{2} / L_{h}^{2}$ for $z \geq 0$, to follow (very roughly) data for a 14.4 GHz ECRIS[2].

Cylindrical symmetry makes components with different azimuthal wavenumbers $m$ uncoupled, so that base functions are

$$
\begin{equation*}
E_{r}=\left[E_{r}^{S}(r, z) \sin (m \vartheta)+E_{r}^{c}(r, z) \cos (m \vartheta)\right] \mathrm{e}^{\mathrm{i} \omega t} \tag{11}
\end{equation*}
$$

and similarly for $E_{\vartheta}$ and $E_{z}$; in the following $m=1$ and $k_{z}$, $\beta_{o}$ are the (positive) numeric solutions of

$$
\begin{equation*}
\frac{\omega^{2}}{c^{2}}=k_{r}^{2}+k_{z}^{2}=\frac{3.8317^{2}}{b^{2}}-\beta_{o}^{2} \quad, \quad k_{r}=\frac{1.8412}{b} \tag{12}
\end{equation*}
$$

within shown precision. The input wave is $\propto \exp \left(-i k_{z} z\right)$ and the reflected wave is $\propto \exp \left(i k_{z} z\right)$, so that
$a^{+}(r, z) \equiv \mathrm{e}^{\mathrm{i} k_{z} z}\left[a(r, z)+\left(\mathrm{i} / k_{z}\right) a_{, z}\right] \quad, \quad a \in\left\{E_{r}^{s}, E_{r}^{c}, E_{\vartheta}^{S}, E_{\vartheta}^{c}\right\}$


Figure 3. Maps of $E_{z}$ field components.


Figure 4. Maps of leading fields.
selects the forward wave projected at $z=0$ (and $a^{-}$selects the backward wave). We choose $y$-polarization for the launched $\mathrm{TE}_{11}$ wave, so that[6]

$$
\begin{equation*}
E_{r}^{S+}=2 m J_{m}\left(k_{r} r\right) /\left(k_{r} r\right) \quad, \quad E_{\vartheta}^{c+}=2 J_{m}^{\prime}\left(k_{r} r\right) \tag{14}
\end{equation*}
$$

and $E_{r}^{c+}=0=E_{\vartheta}^{S+}$ are the bc at $z=-L_{g}$, where input amplitude $E_{y}(r=0)=1 \mathrm{~V} / \mathrm{m}$ is understood. Moreover since $E_{z} \propto \mathrm{e}^{\beta_{o} z}$ is there evanescent, we get the other bc $E_{z, z}^{c}=\beta_{o} E_{z}^{c}$ and $E_{z, z}^{s}=\beta_{o} E_{z}^{s}$.

Thanks to eq. (11), numerical solution can be reduced to a 2D geometry, for the six components with $m=1$, in perspective allowing a very refined mesh (now on the exposed corner mesh size is $h_{c} \cong 10^{-5} \mathrm{~m}$ ). To use a standard PDE interface [8] which gives full control on boundary conditions, some rearrangements of eq. (2) were performed, mainly grouping terms in the $r$ curl curl $\mathbf{E}$ expression.

## 4 Results

The field pattern of eq. (2) solutions is very rich and complicate, and some selection of figures is necessary. We start from $v / \omega=0.05$, since lower values implies spatially sharper resonances (also called ECR layers) with more numerical effort, and higher values implies a really large collision frequency; firstly we consider a very low density plasma $o_{l} \equiv \max \left(\omega_{p} / \omega\right)=0.01$, so to resemble an empty


Figure 5. Forward and backward waves in $E_{r}^{s}$; eq. (14) level is marked '(in)'.
cavity. Notwithstanding this, some conversion from $\mathrm{TE}_{11}$ to $E_{z}$ appears, as shown in Fig. 3.

The leading components $E_{r}^{s}$ and $E_{\vartheta}^{c}$ are shown in Fig. 4; relative position of maxima of real and imaginary parts are related to wave propagation, which anyway is better seen in Fig. 5, where $E_{r}^{s \pm}$ components are plotted as a function of $z$ for a fixed $r=b / 2$. Inside waveguide traveling wave amplitudes are constant (as they must) while they change in the cavity and at the junction, for effect of plasma interaction, cavity wider radius and junction corners. So most important quantities to monitor among simulations are: the reflection coefficient $R_{e f l}=\left|\mathbf{E}_{\perp}^{-}\right|^{2} / \mid\left. E_{y}($ in $)\right|^{2}$ summing on $r, \vartheta$ and $c, s$ components at input $z=-L_{g}$; the 'conversion to $E_{z}{ }^{\prime}$ coefficient $F_{1}$ defined as $F_{1}=\max _{z \in D}\left(\left|E_{z}^{c}\right|^{2}+\right.$ $\left.\left|E_{z}^{s}\right|^{2}\right)^{1 / 2}(z) / \mid E_{y}($ in $) \mid$ where the maximization domain $D$ is $z>0.06 \mathrm{~m}$; and the rotation $\vartheta_{R}$ of reflected waves, defined as $\vartheta_{R}=\operatorname{atan}\left(\left|E_{x}^{-}\right| /\left|E_{y}^{-}\right|\right)$at input. Morevoer, $\mid E_{y}($ in $) \mid=$ $\left|\mathbf{E}_{\perp}^{+}\right|$at input and all quantities are calculated on the $r=b / 2$ line for simplicity. In Fig. 4 case $R_{e f l}=0.877$, showing that also a very weak plasma adsorbs $12 \%$ of input power (because wave passes two times in the cavity, that is 4 times through ECR layers); anyway for this weak plasma, rotation $\vartheta_{R}=4.3 \mathrm{mrad}$ and conversion factor $F_{1}=0.0011$ are smaller than for denser plasmas (Fig. 6). About conversion to $E_{z}$ note that input waveguide radius is smaller then required tor $\mathrm{TM}_{10}$ propagation, so that $E_{z}$ can not exit as a backward wave, but it makes a standing wave pattern in the plasma chamber (whose strength is measured by $F_{1}$ as


Figure 6. Sensitivity to collisions: reflection, rotation and $E_{z}$-conversion factor $F_{1}$ vs $v / \omega$ for $o_{l}=\max \omega_{p} / \omega=0.05$


Figure 7. As Fig. 6, but for $o_{l}=\max \omega_{p} / \omega=0.1$


Figure 8. Sensitivity to density: reflection, rotation and $E_{z}$-conversion factor $F_{1}$ vs $o_{l}=\max \omega_{p} / \omega$ for $v / \omega=0.05$
noted); about rotation note that $E_{x}^{-}$and $E_{y}^{-}$have a different time phase, so $\vartheta_{R}$ is a kind of average rotation.

In Fig. 6 plasma density an increase of $n_{e}$ by a factor 25 (as result of factor 5 increase of $o_{l}$ ) allows an increase of wave adsorption, with reflection below $10 \%$ for $v / \omega>0.001$, and reasonable rotation and conversion factors. A further increase of density (see Fig. 7) makes reflection larger again, approaching 1 in the $v / \omega \rightarrow 0$ limit (as it should). To better see the effect of larger densities, Figs. 8 and 9 are scans on $o_{l}=\max \omega_{p} / \omega$ for fixed collision rate $v$. In Fig. 8 with $v / \omega=0.05$ we note: the $\omega_{p}=0$ point corresponding to a truly empty cavity, with reflection $R_{e f l}=1$ as expected; then $R_{e f l}$ sharply decreases approaching zero for $o_{l} \cong 0.06$ where $\vartheta_{R}$ has a maximum (due to the minimum of $E_{y}$ reflection); for $o_{l}>0.2$ it appears a trend of increase of $R_{e f l}$ (with fluctuations) and of $F_{1}$. Figure 9 with $v / \omega=0.1$ is similar, but for $o_{l}>0.2$ we see that $R_{e f l} \cong 0.2$ stabilizes and $F_{1}$ decreases; the reflection minimum for $o_{l}=0.06$ still exists.

In perspective future studies may include: mesh refined on ECR layers[4]; dependence of $v$ from $z$ and improved $K$ models; an output waveguide (or radiation from a hole) in $z=L_{p}$ plane. In conclusion, this model demonstrates that microwave coupling to a magnetized plasma can be reliably studied even when cavity size is comparable to wavelength;


Figure 9. As fig. 8, but for $v / \omega=0.1$
the several approximations involved with setting a 2 D geometry model may help theoretical understanding, on the ECR plasmas as well as on related plasma waves. Moreover the 2D model has obvious advantages for parametric scans and improving mesh accuracy. As preliminary physical results, the complete adsorption of microwave was found possible even for small collision rates and plasma densities, while higher densities may show some mismatch (perhaps to be tuned with magnetic field profile or more realistic geometries).

## References

[1] R.Geller, Electron Cyclotron Resonance Ion Sources and ECR Plasmas, (1996); IOP Publishing, Bristol; especially chapter 2.
[2] M. Cavenago, "Modelling microwave conversion to longitudinal waves in small ECR plasmas", Rev. Sci. Instrum., 67, pp. 1079-1081 (1996).
[3] F. Wenander, J. Lettry, and the ISOLDE Collaboration, "MECRIS: A compact ECRIS for ionization of noble gas radioisotopes at ISOLDE," Rev. Sci. Instrum. 75, 1627 (2004); doi: 10.1063/1.1691473.
[4] G. Torrisi, D. Mascali, A. Galatà, et al., "Plasma heating and innovative microwave launching in ECRIS: models and experiments," $J$ Instrum, 14, 2019, C01004, and references therewithin, doi:10.1088/1748-0221/14/01/C01004.
[5] G. D. Swanson, Plasma Waves, (2003), 2nd ed. 2003, IOP Publishing, Bristol; especially p 37 and 118.
[6] C. A. Balanis, Advanced engineering electromagnetics (2012) 2nd ed. J. Wiley, New York; especially chapter 9.
[7] E. V. Appleton (1928) URSI Proceedings, Washington Assembly.
[8] Comsol Multiphysics 3.5 (2009) or higher versions, see also http://www.comsol.eu

