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Asymptotic Model of Acoustic Scattering by Thin Elastic Shells in Fluids

Evgeny V. Chernokozhin* (1) and Amir Boag(1)
(1) Tel Aviv University, Tel Aviv 69978, Israel, e-mail: evg.chernokozhin@gmail.com; boag@tau.ac.il

Abstract

An asymptotic model of acoustic scattering by a thin elastic shell is developed. The shell is replaced with effective boundary conditions involving surface differential operators. The shell thickness is considered a small parameter. Analytical solutions within asymptotic approximations in the spherical case are compared with the exact solution for an elastic spherical layer. Resonance effects of the first and third order in the shell thickness are distinguished.

1 Introduction

The full rigorous problem formulation of acoustic scattering by elastic bodies in fluids conventionally involves scalar wave equations for the pressure in the fluids with tensor equations for the stresses in the elastic bodies. These equations are coupled via boundary conditions at the fluid-elastic body interfaces. Such formulations results in complicated boundary value problems that are very difficult to solve efficiently, because of their three-dimensional tensor nature.

In an important special case when the elastic body is a thin shell, the problem can be significantly simplified by replacing the shell with effective boundary conditions in which the shell is treated as infinitely thin and all its physical properties, including its actual thickness, enter into the boundary conditions. It is assumed that the shell thickness h is much smaller than the wavelengths in the fluid and the shell material and is much smaller that the curvature radii of the shell; thus, h is considered a small parameter. Under these assumptions, the pressure jump across the shell is expressed through the average normal displacement and the difference of the boundary displacements is expressed through the average boundary pressure. Thus, we avoid solving a 3D tensor problem of elastoacoustics in the shell and, instead, consider a scalar boundary value problem for the pressure satisfying wave equation(s). This opens up opportunities for developing efficient numerical methods [1].

2 Problem statement

Consider an elastic shell of thickness h, immersed in a fluid. The shell encloses a region filled with the same or a different fluid or, possibly, with vacuum (Figure 1). The shell is illuminated with a harmonic pressure wave p^{inc} with a time dependence $e^{i\omega t}$. The task is to find the scattered pressure p^{sc}.

The scattered pressure p^{sc} and the pressure p^{1} in the interior of the shell are described by a system of two Helmholtz equations:

$$\begin{aligned} &\nabla^2 p^{\text{sc}} + k_+^2 p^{\text{sc}} = 0 \\ &\nabla^2 p^{1} + k_-^2 p^{1} = 0 \end{aligned} \tag{1}$$

where k_+ and k_- are, generally speaking, different wavenumbers.

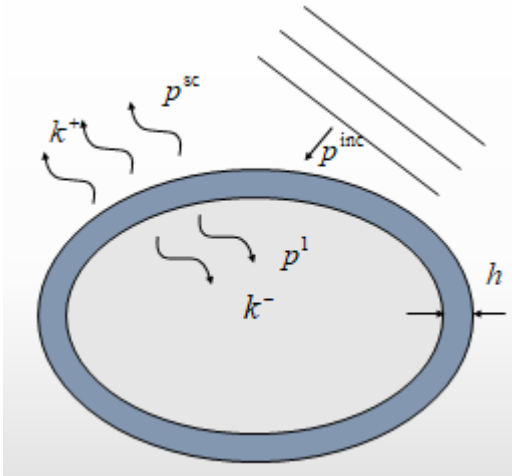

Figure 1. Geometry of the problem comprising a shell of thickness h immersed in a fluid and filled with a different fluid illuminatted by a plane wave.

In the shell, instead of the scalar pressure field, we have a tensor field of stresses σ_{ij}. The stresses are expresses via strains, and the strains are expressed via 3D displacements $\mathbf{u}$. The displacements are described by the Navier equation

$$\nabla\nabla\cdot\mathbf{u} - q^2\nabla\times(\nabla\times\mathbf{u}) + k_L^2\mathbf{u} = 0\,,$$

where $q^2 = k_L^2/k_T^2$ is the squared ratio of the longitudinal, k_L, and transverse, k_T, wavenumbers in the shell material, respectively.

To link the wave equations, we add boundary conditions: 1) the continuity conditions, equating the normal displacements in the fluid, $w_n^{\pm}$, and in the shell, $u_n^{\pm}$:

$$u_n^{\pm} = w_n^{\pm},$$

2) the equality of the normal stresses to the boundary pressure

$$\sigma_{nn}^{\pm} = -p^{\pm},$$

and 3) the equality of the boundary shear stresses to zero:

$$\sigma_{nj}^{\pm} = 0, \quad j = 2, 3.$$

3 Effective boundary conditions

If the shell thickness is small compared to the linear sizes of the shell and the wavelengths in the fluids and in the shell material, it would be impractical to treat the shell as a volumetric elastic object. Instead, we will regard it as a 2D object and replace it with effective boundary conditions, considering the shell thickness h a small parameter. To that end, we introduce the midsurface S of the shell and introduce on it (locally or globally) a curvilinear coordinate system (q_1, q_2, q_3). The first coordinate is normal to the midsurface: $q_1 = n$. We expand the displacements and stresses at the midsurface in powers of q_1, which gives us expansions of the corresponding boundary values in powers of h:

$$u_n^{\pm}(q_2, q_3) = u_n(0, q_2, q_3) \pm \frac{h}{2} \left. \frac{\partial u_n}{\partial q_1} \right|_S + \ldots,$$

$$\sigma_{nj}^{\pm}(q_2, q_3) = \sigma_{nj}(0, q_2, q_3) \pm \frac{h}{2} \left. \frac{\partial \sigma_{nj}}{\partial q_1} \right|_S + \ldots, \; j = 1, 2, 3.$$

We retain the terms up to the first or third order. The order of the asymptotic model is understood as the maximum power of h in the expansions. The stresses are expressed in terms of the displacements and their derivatives. The higher order derivatives are expresses via lower order derivatives using the Navier equations.

If we additionally assume that the shell thickness is much smaller than the curvature radii of the shell, in the first-order model, we obtain the following boundary conditions, expressing the pressure jumps and the difference of displacements across the surface via the average displacement $\overline{w} = (w^+ + w^-)/2$ and the average pressure $\overline{p} = (p^+ + p^-)/2$:

$$p^+ - p^- = 2h\omega^2 \rho \overline{w}, \tag{1}$$

$$\left(k_T^4 + 4\nabla_S^2 \left(k_T^2 - k_L^2\right)\right)\left(w^+ - w^-\right) = -\frac{2hk_T^4}{\omega^2 \rho}\left(\nabla_S^2 + k_L^2\right)\overline{p}, \tag{2}$$

where ∇_S^2 is the surface Laplacian on the midsurface S and ρ is the mass density of the shell.

In the third-order model, algebraic condition (1) is replaced with the fourth-order surface differential equation

$$\begin{aligned} p^+ - p^- &= 2\omega^2 \rho h \overline{w} + \frac{2h^3\omega^2\rho}{3} k_L^2 \overline{w} - \\ &-2h^3\omega^2\rho\left(1 - \frac{4}{3}q^2\right)\nabla_S^2 \overline{w} - \frac{8h^3\omega^2\rho}{3k_T^2}\left(1 - q^2\right)\nabla_S^4 \overline{w} \end{aligned} \tag{3}$$

Note that, unless the frequency is close to the resonance frequencies of the operator on the left-hand side of (2), this condition in many cases can be replaced with a simpler condition

$$w^+ - w^- = 0, \tag{4}$$

which means that the shell thickness is constant.

The first-order model with conditions (1) and (4) is often called the inertial model, since it neglects the elastic effects.

4 Comparison with the exact solution for a spherical elastic layer

The quality of the asymptotic approximations can be assessed by a comparison with the exact solution for an elastic spherical layer [2]: the only known nontrivial solution of the problem under consideration. In this case, the solution can be expanded in spherical harmonics, thus reducing the boundary value problem to a system of ordinary differential equations, which can be solved explicitly in Bessel functions. The same applies to the asymptotic model, but its solutions are significantly simpler. Nevertheless, in many cases, the exact and asymptotic models give close results.

Figure 2 shows the target strength (TS) of a large steel shell immersed in water and filled with water in the frequency range of 1–10 kHz, calculated within the model of an elastic spherical layer [2], within the 1st-order model with boundary conditions (1) and (2), and within the 3rd-order model with boundary conditions (3) and (4). One can see that the behavior of the curves is the same, although there are some deviations.

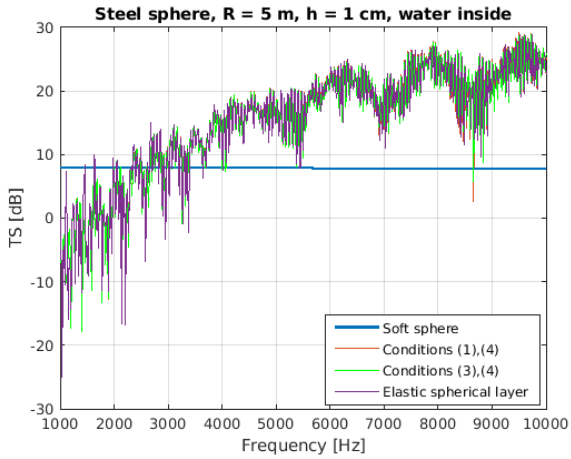

Figure 2. Target strength of a flooded steel sphere, calculated by the exact model for an elastic spherical layer and in the asymptotic approximations with different boundary conditions. For comparison, the horizontal line represents the TS of a soft sphere of the same radius.

5 First- and third-order resonances

Figure 3 shows the target strength of a large and relatively thick vacuum-filled steel sphere immersed in water, calculated by different models. The straight line represents the TS of a soft sphere of the same radius. The TS calculated by the inertial model is close to the same straight line. The exact solution (for an elastic spherical layer) consists of an almost straight line with narrow resonance peaks and with a peculiar resonance curve in the range 5–7 kHz. The first-order elastic approximation reproduces well the narrow resonance peaks. One can see that this model has almost the same narrow low-frequency resonance peaks, but it does not reproduce the high-frequency resonance curve. The third-order approximation with conditions (3) and (4) does not have the low frequency resonances, but reproduces the shape of the high-frequency resonance curve although with some frequency shift. Thus, the low-frequency and high-frequency resonances in Figure 3 may be called the 1st- and 3rd-order resonances of the elastic spherical shell, in accordance with the order of approximation of the asymptotic model in which they are reproduced.

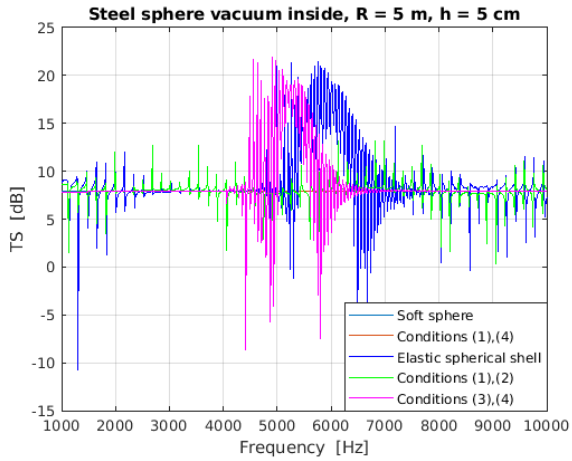

Figure 3. Target strength of a thick vacuum-filled steel sphere, calculated by the exact model for an elastic spherical layer and in the asymptotic approximations with different boundary conditions. For comparison, the horizontal line represents the TS of a soft sphere of the same radius.

6 Use in numerical models

The asymptotic approximations with effective boundary conditions (1)–(4) and others are used to develop efficient numerical models for acoustic scattering by thin shells. In particular, conditions (1) and (4) are used in the problem formulation for a fast iterative solver devised for solving problems with fairly large shells [1]. The development of iterative and direct fast solvers involving conditions (2) and (3) is currently underway. The comparison with analytical solutions is indispensable for validation of the numerical models.

7 References

1. E. Chernokozhin, A. Tsinovoy, and A. Boag, “Scattering by Thin Shells in Fluids: Fast Solver and Experimental Validation,” *JASA Express Lett.*, **1**, 1, January 2021, pp. 016002-1-7, doi: 10.1121/10.0002999.

2. R. R. Goodman and R. Stern, “Reflection and Transmission of Sound by Elastic Spherical Shells,” *J. Acoust. Soc. Am.*, **34**, 3, March 1962, pp. 338-344.