# A computational approach for the inverse problem of reconstructing a spherically symmetric refractive index using modified transmission eigenvalues 

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#### Abstract

The modified transmission eigenvalue problem is a special eigenvalue problem that arises in inverse scattering of inhomogeneous media. We confine to the case where the medium is spherically symmetric and address the inverse problem by introducing computational methods to estimate the unknown refractive index of the medium, from the knowledge of a subset of the spectrum.


## 1 Introduction

Measured scattering data including transmission eigenvalues are important for the determination of unknown parameters of a medium and for applications such as nondestructive testing of materials. More specifically, the transmission eigenvalue problem is indicated as one of the most challenging subjects of inverse scattering theory for inhomogeneous media and has been studied in many directions lately (see e.g. [2, 3, 4] and the references therein).

Although, transmission eigenvalues are determined from the properties of the scatterer and their dependence on the interrogating wave frequencies needed for evaluation, makes them difficult to use. As of recently, new eigenvalue problems have been introduced, in an effort to bypass this obstacle. The main idea is to fix the wave number and use new artificial parameters that in turn play the role of the spectral parameter for the scattering problem. One approach, is to artificially embed the medium into another inhomogeneous medium, which leads to a modified far field operator [1]. The corresponding eigenvalues, namely the modified transmission eigenvalues, can be determined from scattering data and it is shown that they carry information about the scattering object $[1,5,8]$.

The modified transmission eigenvalue problem with an artificial metamaterial background, has the following formulation: find $\lambda \in \mathbb{C}$ such that there is
a non trivial pair of solutions $(w, v)$ for the system

$$
\begin{array}{rlrl}
\Delta w+k^{2} n(x) w & =0, & & x \in D_{b} \\
(-a) \Delta v+k^{2} \lambda n_{0}(x) v & =0, & & x \in D_{b} \\
w & =v, & & x \in \partial D_{b} \\
\frac{\partial w}{\partial v}=(-a) \frac{\partial v}{\partial v}, & & x \in \partial D_{b} \tag{4}
\end{array}
$$

where $D_{b} \subset \mathbb{R}^{m}$ is a bounded domain with Lipschitz boundary $\partial D_{b}, k>0$ is the fixed wave number, $n \in$ $L^{\infty}\left(D_{b}\right)$ is the refractive index, $n_{0} \in L^{\infty}\left(D_{b}\right)$ is the artificial refractive index, $v$ is the outward unit normal to $\partial D_{b}$ and $a>0$. Assuming that $\mathfrak{R}(n), \mathfrak{R}\left(n_{0}\right)>0$ and $\mathfrak{I}(n), \mathfrak{I}\left(n_{0}\right)=0$, secures that the metamaterial refractive index $-n_{0}(x) / a$ is real and negative valued.

In the classic transmission eigenvalue problem, the wave number coincides with the spectral parameter and therefore multifrequency data are required for eigenvalues determination. For the modified problem (1)-(4), $\lambda$ plays the role of the spectral parameter and the wave number is fixed. Thus, a single wave number suffices for the measurement of $\lambda$ from scattering data.

In [8], we proposed a spectral Galerkin method to approximate the modified transmission eigenvalues in general domains. In this paper, we apply this method to spherically symmetric layered media with piecewise constant refractive index. We consider the corresponding inverse problem, to estimate the refractive index from the knowledge of a subset of the spectrum. Firstly, we address the inverse problem by using the few larger eigenvalues to reconstruct a refractive index with two layers. Next, we define a Newton-type inversion scheme for the general spherically stratified domain. Our approach is based on the computational methods of the inverse transmission eigenvalue problem considered in [9, 10]. For the purposes of the present work, we use modified instead of classic transmission eigenvalues.

## 2 Computational methods for the inverse eigenvalue problem

Our aim is to provide numerical evidence that modified transmission eigenvalues carry information about the unknown refractive index. We restrict our analysis in the spherically symmetric case, assuming that $D_{b}$ is the unit disc $B(0.1)$ of $\mathbb{R}^{2}$ and $a, n_{0}$ are positive constants.

### 2.1 The inverse problem for a domain with two layers

We choose our domain to be a unit disc with two layers and we assume that the corresponding refractive index is a piecewise constant function:

$$
n(r):= \begin{cases}n_{1} & 0<r<r_{1} \\ n_{2} & r_{1}<r<1\end{cases}
$$

which is discontinuous at $r=r_{1}$. For this simple case, we can analytically compute the modified transmission eigenvalues, by using separation of variables and applying the transmission and continuity conditions in the boundaries. The corresponding eigenfunctions of (1)-(4) have the following form:
$v_{m}(r, \theta)=a_{m} J_{m}\left(k \sqrt{\frac{\lambda n_{0}}{-a}} r\right) e^{i m \theta}$
$w_{m}(r, \theta)=\left\{\begin{array}{l}b_{m} J_{m}\left(k \sqrt{n_{1}} r\right) e^{i m \theta} \\ \left(c_{m} J_{m}\left(k \sqrt{n_{2}} r\right)+d_{m} N_{m}\left(k \sqrt{n_{2}} r\right)\right) e^{i m \theta}\end{array}\right.$
for $m=0,1, \ldots$, where $J_{m}$ and $N_{m}$ are Bessel and Neumann functions, respectively. Therefore, $\lambda$ is a modified transmission eigenvalue if and only if is a zero of the determinant:

$$
\operatorname{det}\left(\begin{array}{cccc}
J_{m}\left(k \sqrt{\frac{n_{0}}{-a}}\right) & 0 & -J_{m}\left(k \sqrt{n_{2}}\right) & -N_{m}\left(k \sqrt{n_{2}}\right)  \tag{5}\\
\left.(-a) \frac{\mathrm{d}}{} J_{m}\left(k \sqrt{\frac{\lambda n_{0}}{-a}} r\right)\right|_{r=1} & 0 & -\left.\frac{\mathrm{d}}{\mathrm{~d} r} J_{m}\left(k \sqrt{n_{2}} r\right)\right|_{r=1} & -\left.\frac{\mathrm{d}}{\mathrm{~d} r} N_{m}\left(k \sqrt{n_{2}} r\right)\right|_{r=1} \\
0 & J_{m}\left(k \sqrt{n_{1}} r_{1}\right) & -J_{m}\left(k \sqrt{n_{2}} r_{1}\right) & -N_{m}\left(k \sqrt{n_{2}} r_{1}\right) \\
0 & \left.\frac{\mathrm{~d}}{\mathrm{~d} r} J_{m}\left(k \sqrt{n_{1}} r\right)\right|_{r=r_{1}} & -\frac{\left.\mathrm{d} J_{m}\left(k \sqrt{n_{2}} r\right)\right|_{r=r_{1}}}{}-\left.\frac{\mathrm{d}}{\mathrm{~d} r} N_{m}\left(k \sqrt{n_{2}} r\right)\right|_{r=r_{1}}
\end{array}\right)
$$

On the other hand, we use the Galerkin method defined in [8], to numerically approximate the eigenvalues. A weak solution of (1)-(4) defined on $B(0,1)$, is a function pair $(w, v)$ that solves the following equation:

$$
\begin{align*}
& \int_{B} \nabla w \cdot \nabla \overline{w^{\prime}} d x+a \int_{B} \nabla v \cdot \nabla \overline{v^{\prime}} d x  \tag{6}\\
& -k^{2} \int_{B} n(x) w \cdot \overline{w^{\prime}} d x+k^{2} \lambda \int_{B} n_{0} v \cdot \overline{v^{\prime}} d x=0
\end{align*}
$$

for all $\left(w^{\prime}, v^{\prime}\right) \in \mathscr{H}(B):=\left\{(f, g) \in H^{1}(B) \times H^{1}(B)\right.$ : $f=g$ on $\partial B\}$. For the Galerkin approximation
scheme, we choose an appropriate N -dimensional orthonormal system $\left\{\left(\phi_{i}, \psi_{i}\right)\right\}_{i=1}^{N}$ in $\mathscr{H}(B)$. Then, the discrete generalized eigenvalue problem for (6) follows:

$$
\begin{equation*}
M_{1}^{(N)} \mathbf{c}=-\lambda^{(N)} M_{2}^{(N)} \mathbf{c} \tag{7}
\end{equation*}
$$

where $M_{1}, M_{2}$ are the $N \times N$ matrices:

$$
\begin{align*}
M_{1}^{(N)}:= & \int_{B} \nabla \phi_{i} \cdot \nabla \overline{\phi_{j}} \mathrm{~d} x+a \int_{B} \nabla \psi_{i} \cdot \nabla \overline{\psi_{j}} \mathrm{~d} x \\
& -k^{2} \int_{B} n(x) \phi_{i} \overline{\phi_{j}} \mathrm{~d} x  \tag{8}\\
M_{2}^{(N)}:= & k^{2} \int_{B} n_{0} \psi_{i} \overline{\psi_{j}} \mathrm{~d} x
\end{align*}
$$

and $\mathbf{c}=\left(c_{1}, c_{2}, \ldots, c_{N}\right)^{T} \in \mathbb{R}^{N}, \quad i, j=1,2, \ldots, N$. In [8], it is shown that the approximate eigenvalues $\lambda^{(N)}$ converge to the eigenvalues of (1)-(4).

The idea for solving the inverse problem, is to minimize the relative percent error between the first $m$ computed and analytically known eigenvalues

$$
\begin{equation*}
f(n):=\sum_{i=1}^{m} \frac{\left|\lambda_{i}^{(N)}(n)-\lambda_{i}\right|}{\left|\lambda_{i}\right|} \% \tag{9}
\end{equation*}
$$

with the piecewise constant index being an unknown. From analytical computations using (5), we noticed that eigenvalues have significant difference in order of magnitude. As a result, we choose to minimize the relative percent error instead of the absolute error. An example of this inversion method, applied for simple case of a constant index, is given in [8].

For the piecewise constant index, we use a basis of 40 eigenfunctions and compute the corresponding $40 \times 40$ matrices defined in (8), for $0.1 \leq r_{1} \leq 1$ with step 0.1. We solve the generalized eigenvalue problem (7) for indices in the range: $0.1 \leq n_{1}, n_{2} \leq 10$ and step 0.1 , using matlab function eig. We create a database of computed eigenvalues $\lambda^{(N)}$ for all possible combinations of ( $n_{1}, n_{2}, r_{1}$ ), and by minimizing the error (9) for the largest $m=4$ eigenvalues, we reconstruct $n(r)$. Some examples are given in Table 1 , where in all cases we have fixed the wave number at $k=0.5$ and the metamaterial parameters at $a=2$ and $n_{0}=1$.

There are cases where the reconstructions are not accurate enough. As a result, we consider the possibility of using modified transmission eigenvalues for multiple wave numbers. This could potentially show that spectra coming from one wave number, are not enough for the determination of an unknown refractive index. In Table 2, we give some examples where we use the largest $m=4$ eigenvalues corresponding to $k=0.5$ and $k=\{0,1,0.2,0.3,0.4,0.5\}$ respectively. When five wave numbers instead of one are

Table 1. Reconstruction of a piecewise constant refractive index from modified transmission eigenvalues

| $\left(n_{1}, n_{2}, r_{1}\right)$ | computed $\lambda_{i}^{(N)}$ | reconstruction |
| :---: | :---: | :---: |
| $(9.2,1.5,0.3)$ | $(2.6398,-36.5788,-98.7148,-115.1933)$ | $(8.5,1.6,0.3)$ |
| $(6,0.3,0.4)$ | $(1.4268,-37.2794,-99.2614,-116.3238)$ | $(5.6,0.4,0.4)$ |
| $(2.1,3.5,0.5)$ | $(3.6369,-35.4089,-97.7453,-114.2539)$ | $(2.7,4,0.8)$ |
| $(3.2,4.8,0.7)$ | $(4.8283,-34.6907,-97.1716,-113.1924)$ | $(3.3,5.4,0.8)$ |
| $(0.2,4,0.9)$ | $(0.9394,-36.6911,-98.5589,-116.7713)$ | $(0.2,4,0.9)$ |

Table 2. Reconstruction of a piecewise constant refractive index from modified transmission eigenvalues corresponding to one and five wave numbers respectively

|  | reconstruction | reconstruction |
| :---: | :---: | :---: |
| $\left(n_{1}, n_{2}, r_{1}\right)$ | $k=0.5$ | $k=\{0,1,0.2,0.3,0.4,0.5\}$ |
| $(9.2,1.5,0.3)$ | $(8.5,1.6,0.3)$ | $(9.2,1.5,0.3)$ |
| $(2.1,3.5,0.5)$ | $(2.7,4.0,0.8)$ | $(2.4,3.4,0.5)$ |
| $(0.2,4.0,0.6)$ | $(1.4,8.1,0.9)$ | $(0.2,4.0,0.6)$ |
| $(3.2,7.6,0.6)$ | $(4.4,9.1,0.9)$ | $(2.2,7.3,0.5)$ |
| $(7.5,1.2,0.2)$ | $(3.4,1.6,0.3)$ | $(7.5,1.5,0.2)$ |

used, the reconstructions are significantly improved. We also tested our computational method by adding $0.1 \%$ error to the eigenvalues and the reconstructions were successful as well. From the above examples, we see that the few largest modified transmission eigenvalues provide information about a piecewise constant refractive index, including the unknown position of the discontinuity. As a result, this method can be useful in applications like non destructive testing of materials with two layers.

### 2.2 A Newton-type scheme for the general spherically stratified domain

We now define a Newton-type method, to estimate a general piecewise constant index from a subset of the spectrum. In contrast with the previous inversion scheme, we do not have to pair original and computed eigenvalues and minimize their relative error to reconstruct the unknown index.

We assume that $B$ is a unit disc with $L$-layers such that $B=\cup_{i=1}^{L} B_{i}$ and the corresponding boundaries $\left\{\partial B_{i}\right\}_{i=1}^{L}$ are concentric circles. The picewise constant refractive index is given by:

$$
n(r):= \begin{cases}n_{1} & r \in B_{1} \\ \vdots & \\ n_{L} & r \in B_{L}\end{cases}
$$

The following Newton method, was initially developed for inverse mass-spring vibrating systems in
[7]. Authors of [9, 10], adapted that method to the inverse quadratic eigenvalue problem for the classic transmission eigenvalues. In this work, we modify the algorithm and apply it to the inverse generalized eigenvalue problem (7), for modified transmission eigenvalues.

We rewrite matrices (8) in the following form:

$$
\begin{align*}
& M_{1}^{(N)}:=\nabla \Phi^{(N)}+a \nabla \Psi^{(N)}-k^{2} \sum_{l=1}^{L} n_{l} \Phi_{l}^{(N)}  \tag{10}\\
& M_{2}^{(N)}:=k^{2} n_{0} \Psi^{(N)}
\end{align*}
$$

where
$\nabla \Phi^{(N)}=\int_{B} \nabla \phi_{i} \cdot \nabla \overline{\phi_{j}} \mathrm{~d} x, \quad \nabla \Psi^{(N)}=\int_{B} \nabla \psi_{i} \cdot \nabla \overline{\psi_{j}} \mathrm{~d} x$ $\Phi_{l}^{(N)}=\int_{B_{l}} \phi_{i} \overline{\phi_{j}} \mathrm{~d} x, \quad \Psi^{(N)}=\int_{B} \psi_{i} \overline{\psi_{j}} \mathrm{~d} x$
for $i, j=1,2, \ldots, N$ and $n_{0}$ being a constant. The inverse spectral problem is given a set of eigenvalues $S=\left\{\mu_{i}\right\}_{i=1}^{N}$ to determine the scalars $\left\{n_{l}\right\}_{l=1}^{L}$ such that $P(\lambda)=M_{1}^{(N)}+\lambda M_{2}^{(N)}$ has spectrum $\sigma(P(\lambda))=$ $S$.

The Newton method seeks a vector $n=\left(n_{1}, \ldots, n_{L}\right)$ which solves the nonlinear system $f(n)=$ $\left.\left(f_{1}(n), \ldots, f_{N}(n)\right)\right)^{\top}=(0, \ldots, 0)^{\top}$, where:

$$
\begin{aligned}
f_{i}(n):= & \operatorname{det}\left[\nabla \Phi^{(N)}+a \nabla \Psi^{(N)}-k^{2} \sum_{l=1}^{L} n_{l} \Phi_{l}^{(N)}\right. \\
& \left.+\mu_{i}\left(k^{2} n_{0} \Psi^{(N)}\right)\right]
\end{aligned}
$$

In cases where the number of layers $L$ is smaller that the number of eigenvalues, we have an overdetermined system of equations and the corresponding method is a Gauss-Newton iterative method for an unconstrained minimization problem [6].

### 2.2.1 The algorithm

We describe the main steps of the Newton-type iteration method:

## Input

- the artificial metamaterial parameters $a, n_{0}$ and the fixed wave number $k$
- the set of $\nabla \Phi^{(N)}, \nabla \Psi^{(N)},\left\{\Phi_{l}^{(N)}\right\}_{l=1}^{L}$ and $\Psi^{(N)}$ $N \times N$ matrices
- an initial estimate of $n^{(0)}=\left(n_{1}^{(0)}, n_{2}^{(0)}, \cdots, n_{L}^{(0)}\right)$ of the unknown refractive index $\left\{n_{l}\right\}_{l=1}^{L}$
- the set $S=\left\{\mu_{i}\right\}_{i=1}^{N}$ of modified transmission eigenvalues

Table 3. Reconstruction of a piecewise constant refractive index with five layers with the Newton method

| $\left(n_{1}, n_{2}, n_{3}, n_{4}, n_{5}\right)$ | initial guess | reconstruction | steps |
| :---: | :---: | :---: | :---: |
| $(2.3,6.7,8.5,4.1,3.0)$ | $(5,5,5,5,5)$ | $(2.30,6.69,8.50,4.09,2.99)$ | 14 |
| $(9.0,7.0,3.0,8.0,2.0)$ | $(6,6,6,6,6)$ | $(8.99,7.00,2.99,7.99,2.00)$ | 12 |
| $(0.5,7.2,7.2,0.3,0.3)$ | $(4,4,4,4,4)$ | $(0.49,7.20,7.19,0.29,0.30)$ | 16 |
| $(0.2,0.2,0.2,4.0,4.0)$ | $(2,2,2,2,2)$ | $(0.19,0.20,0.19,3.99,4.00)$ | 14 |
| $(6.0,6.0,0.3,0.3,0.3)$ | $(2,2,2,2,2)$ | $(5.99,6.00,0.29,0.29,0.30)$ | 14 |

## Output

A vector of the estimated $\left\{n_{l}\right\}_{l=1}^{L}$ which are such that $\sigma(P(\lambda))=S$

## The Iteration

1. Choose a starting value $n^{(0)}$ for the vector of the unknown coefficients
2. for $s=0,1, \cdots$
(a) compute the Jacobian $J\left(n^{(s)}\right)$ and the function $f\left(n^{(s)}\right)$
(b) solve the system:
$J\left(n^{(s)}\right) \xi^{(s)}=-f\left(n^{(s)}\right)$, for $\xi^{(s)}$
(c) compute the new estimate of the coefficients vector $n^{(s+1)}=\xi^{(s)}+n^{(s)}$
(d) stop when $\left\|\xi^{(s)}\right\|$ smaller that desired tolerance
end loop (ii) ( $s$-loop)

The Jacobian matrix in the above iteration scheme is computed using a QZ - generalized Schur decomposition. We refer to [7, 9, 10] for more details.

We test the algorithm for the case a of unit disc with five layers and width 0.2 for each layer. We use a system with 20 elements and compute the corresponding matrices (10). Using matlab function eig, we calculate the highest 8 eigenvalues, which in turn are the input eigenvalues for the inversion scheme. In all cases we fix $k=0.5, a=2$ and $n_{0}=1$, in a similar fashion with the previous method. We note that for the inverse problem, the position of each layer is not a priori known. Some examples are presented in Table 3.

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