# Fast Converging Technique for the Analysis of Electromagnetic Scattering from a Zero-Thickness PEC Disk in a Layered Medium

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## Abstract

In this paper, the analysis of the electromagnetic scattering from a zero-thickness perfectly electrically conducting disk is carried out by means of the method of analytical preconditioning applied to an integral formulation in the vector Hankel transform domain. A complete set of orthogonal eigenfunctions of the static part of the integral operator reconstructing the physical behavior of the surface current density is used to discretize the integral equation. In this way, the obtained matrix equation, which is a Fredholm second-kind equation, is fast convergent. Numerical results are provided showing the efficiency of the proposed method.

## 1 Introduction

When searching for the solution of an electromagnetic problem, integral equation formulations taking into account the radiation condition for the fields, associated with discretization techniques are widely used, where the integral equations and the unknowns are defined on finite support. The problem of the existence of a solution of an arbitrary integral equation and, if such a solution exists, of the convergence of an arbitrary discretization scheme is a key point because the Fredholm theory can be applied only if the operator is the superposition of a continuously invertible operator and a completely continuous operator [1].

A class of methods associated with analytical regularization is aimed at converting general integral equations to integral or matrix equations for which the Fredholm theory is valid [2]. A Fredholm second-kind integral equation can be obtained by analytically inverting the most singular part of the integral operator [3, 4]. Hence, the convergence of any direct discretization that keeps Fredholm's nature is guaranteed. On the other hand, guaranteed convergence is achieved by means of a complete set of orthogonal eigenfunctions of a suitable operator containing the most singular part of the original integral operator as expansion basis in a fullwave meshless Galerkin scheme [5-10]. In that case, Galerkin-projection technique acts as a perfect analytical preconditioner for the considered integral equation.

In this paper, the method of analytical preconditioning is successfully applied to the electromagnetic scattering from a zero-thickness perfectly electrically conducting (PEC) disk in a planar layered medium. The revolution symmetry of the problem allows to expand all the involved functions in terms of the series of orthogonal cylindrical harmonics. Hence, the problem is conveniently cast to a set of integral equations for the vector Hankel transform of the harmonics of the surface curl-free and surface divergence-free contributions of the surface current density. The obtained integral equations are discretized by using Galerkin method with the expansion functions reconstructing the edge behavior and the behavior around the center of the disk of the corresponding cylindrical harmonic of the surface current density, having a closed form spectral domain counterparts, and forming a set of orthogonal eigenfunctions of the static parts of the associated operators.

## 2 Formulation and Solution of the Problem

A zero-thickness PEC disk is located at the $q$-th interface of a planar layered medium. A cylindrical coordinate system $(\rho,\phi,z)$ with the origin at the center of the disk and the $z$ axis orthogonal to it is introduced. A known incident field $\left(\underline{E}^{inc}(\underline{r}),\underline{H}^{inc}(\underline{r})\right)$, where $\underline{r}=x\hat{x}+y\hat{y}+z\hat{z}$, induces a surface current density $\underline{J}(\rho,\phi)=J_\rho(\rho,\phi)\hat{\rho}+J_\phi(\rho,\phi)\hat{\phi}$ on the disk that, in turn, generates a scattered field such that the tangential component of the total electric field vanishes on the disk surface and the fields satisfy the edge conditions and the radiation conditions. Under these conditions, the search for $\underline{J}(\rho,\phi)$ has unique solution.

On the other hand, the revolution symmetry of the problem allows to expand all the involved functions in series of orthogonal cylindrical harmonics. Hence, the problem can be equivalently reduced to an infinite set of independent one-dimensional equations obtained by demanding each harmonic of the total electric field to be vanishing on the disk surface, i.e., [10]

$$\int_0^{+\infty} \underline{\underline{\mathbf{H}}}^{(n)}(w\rho)\underline{\underline{\tilde{\mathbf{G}}}}_q(w)\underline{\tilde{\mathbf{J}}}^{(n)}(w)\,w\,dw = -\underline{\mathbf{E}}^{inc(n)}(\rho,0) \tag{1}$$

for $\rho \le a$, where the superscript $n$ is harmonic's number,

$$\underline{\tilde{\mathbf{J}}}^{(n)}(w) = \int_0^{+\infty} \underline{\underline{\mathbf{H}}}^{(n)}(w\rho)\underline{\mathbf{J}}^{(n)}(\rho)\rho\, d\rho \tag{2}$$

is the vector Hankel transform of order $n$ ($\mathrm{VHT}_n$) of

$$\underline{\mathbf{J}}^{(n)}(\rho) = \begin{pmatrix} J_\rho^{(n)}(\rho) \\ -jJ_\phi^{(n)}(\rho) \end{pmatrix}, \tag{3a}$$

$$\underline{\underline{\mathbf{H}}}^{(n)}(w\rho) = \begin{pmatrix} J_n'(w\rho) & nJ_n(w\rho)/(w\rho) \\ nJ_n(w\rho)/(w\rho) & J_n'(w\rho) \end{pmatrix}, \tag{3b}$$

where $J_n(\cdot)$ and $J_n'(\cdot)$ are the Bessel function of the first kind and order $n$ and its first derivative with respect to the argument, respectively [11],

$$\underline{\underline{\tilde{\mathbf{G}}}}_q(w) = \begin{pmatrix} \tilde{G}_{q,C}(w) & 0 \\ 0 & \tilde{G}_{q,D}(w) \end{pmatrix} \tag{4}$$

is the spectral domain Green function [12, 13], and

$$\underline{\mathbf{E}}^{inc(n)}(\rho,0) = \begin{pmatrix} E_\rho^{inc(n)}(\rho,0) \\ -jE_\phi^{inc(n)}(\rho,0) \end{pmatrix}. \tag{5}$$

By means of Helmholtz decomposition, the $n$-th harmonic of the surface current density can be represented as the superposition of a surface curl-free contribution

$$\underline{\mathbf{J}}_C^{(n)}(\rho) = \begin{pmatrix} \frac{d}{d\rho} \\ \frac{n}{\rho} \end{pmatrix} \Phi_C^{(n)}(\rho), \tag{6}$$

and a surface divergence-free contribution

$$\underline{\mathbf{J}}_D^{(n)}(\rho) = -j\begin{pmatrix} \frac{n}{\rho} \\ \frac{d}{d\rho} \end{pmatrix} \Phi_D^{(n)}(\rho), \tag{7}$$

where the functions $\Phi_T^{(n)}(\rho)$ for $T \in \{C, D\}$ are suitable potential functions [14]. It is simple to observe that the $\mathrm{VHT}_n$ of $\underline{\mathbf{J}}_T^{(n)}(\rho)$ have only one non-vanishing component, i.e.,

$$\underline{\tilde{\mathbf{J}}}_C^{(n)}(w) = \begin{pmatrix} \tilde{J}_C^{(n)}(w) \\ 0 \end{pmatrix}, \tag{8a}$$

$$\underline{\tilde{\mathbf{J}}}_D^{(n)}(w) = \begin{pmatrix} 0 \\ -j\tilde{J}_D^{(n)}(w) \end{pmatrix}. \tag{8b}$$

Hence, in order to deal with scalar unknowns in the spectral domain, the surface curl-free and the surface divergence-free contributions of the surface current density are selected as new unknowns.

The obtained integral equations are discretized by means of Galerkin method. The unknown functions in the spectral domain are expanded in a complete series of Bessel functions [15], i.e.,

$$\tilde{J}_T^{(n)}(w) = \sum_{h=-1+\delta_{n,0}}^{+\infty} \gamma_{T,h}^{(n)} \beta_{T,h}^{(n)} \frac{J_{|n|+2h+p_T+1}(aw)}{w^{p_T}}, \tag{9}$$

where $\gamma_{T,h}^{(n)}$ denote the expansion coefficients,

$$\beta_{T,h}^{(n)} = \sqrt{2(|n|+2h+p_T+1)}, \tag{10}$$

$p_C = 3/2$ and $p_D = 1/2$. With such a choice, the integral equations are reduced to Fredholm second-kind matrix operator equations, for which the convergence of the approximate solution of the truncated matrix equation to the exact solution of the problem, as the truncation order tends to infinity can be established for any $n$. Moreover, the behavior of the $n$-th harmonic of the surface current density at the edge and around the center of the disk is correctly reconstructed leading to a fast convergence and the convolution integrals resulting from Galerkin-projection technique are automatically reduced to algebraic products.

## 3 Numerical Results

As mentioned, approximate solution can be obtained by truncating the obtained infinite matrix equation. In order to show the fast convergence of the presented method, the following normalized truncation error is introduced

$$\mathrm{err}_N(M) = \sqrt{\sum_{n=-N+1}^{N-1} \left\|\mathbf{x}_{M+1}^{(n)} - \mathbf{x}_M^{(n)}\right\|^2 \Big/ \sum_{n=-N+1}^{N-1} \left\|\mathbf{x}_M^{(n)}\right\|^2}, \tag{11}$$

where $2N-1$ is the number of cylindrical harmonics estimated as in [16], $\|\cdot\|$ is the usual Euclidean norm and $\mathbf{x}_M^{(n)}$ is the vector of the expansion coefficients evaluated by using $M$ expansion functions for both the surface curl-free and the surface divergence-free contributions of the $n$-th harmonic of the surface current density.

In figure 1a, the normalized truncation error is plotted as a function of $M$ in the case of TM polarized plane wave, with incident angles $\theta_i = 30\,\mathrm{deg}.$ with the $z$ axis and $\phi_i = 0\,\mathrm{deg}.$ with the $x$ axis in the $xy$ plane, and amplitude $|\underline{E}_0| = 1$ V/m, impinging onto a disk located on the top of an half-space of dielectric permittivity $\varepsilon_r = 3$, for different values of the radius ($a = \lambda/2, \lambda, 2\lambda$). It is clear that the convergence is of exponential type in all the examined cases. For the sake of completeness, in figures 1b and 1c the non-vanishing component of the surface current density and the bistatic radar cross-section (BRCS) for $\phi = 0, 180$ deg. are shown for all the examined cases.

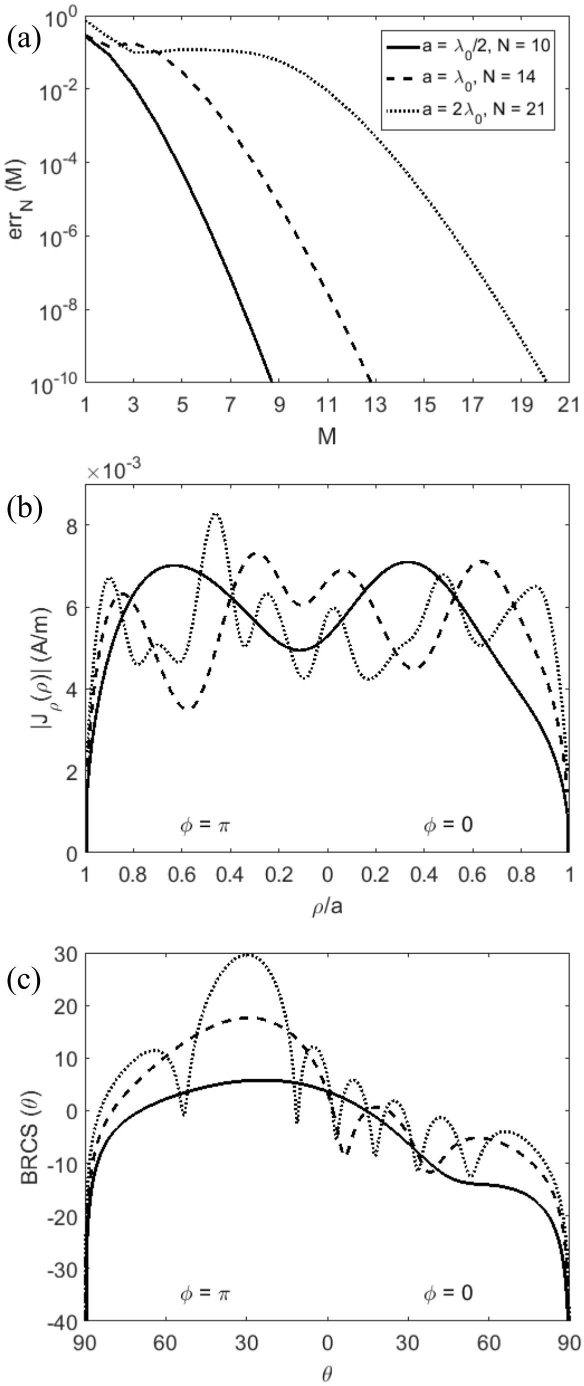


**Figure 1**. (a) Normalized truncation error, (b) surface current density and (c) bistatic radar cross section for disks with different radii. $\theta_i = 30$ deg., $\phi_i = 0$ deg., $|\underline{E}_0| = 1$ V/m and TM polarization.

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