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Comparison of the wave spectra of open and closed inhomogeneous waveguides

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Abstract

The problems are considered of the propagation of surface TE-polarized electromagnetic waves in the Goubau line (perfectly conducting cylinder covered with a concentric dielectric layer) and a shielded two-layer dielectric waveguide filled with an inhomogeneous medium. Numerical results are presented of the comparison of the wave spectra.

1 Introduction

The wave propagation in shielded (closed) waveguide structures with inhomogeneous filling has been an object of intensive studies since the classical work [1] continued in [2, 3] an other studies. The methods appicable for shielded and open waveguides have been developed in [4, 5, 6] using rigorous mathematical approaches, in particular spectral theory of differential and integral operators and operator-valued funstions [7, 8, 9] which was partially reflected in recent monographs [11, 10] on electromagnetic field theory, to name the few.

As far as investigations of spectral properties of such problems is concerned, the methods of nonselfadjoint operator theory [5] turns out to be natural and efficient. After the initial boundary value problem (BVP) has been reduced to the analysis of a certain operator-value function, in many cases operator pencil [5, 12], one can use the apparatus of functional analysis to study the BVP spectral properties (see [12]-[14]). In [12]-[15], a general theory of the propagation of normal waves in closed waveguides was constructed.

Open waveguide structures have been studied by many authors [4, 5, 7, 8, 9] and in [12]-[14] using the methods of spectral theory of nonselfadjoint operators. However, for open (non-shielded) structures, a sufficiently complete theory of wave propagation has not been constructed. In this paper we perform a comparative numerical study of the surface TE-polarized wave propagation in two basic types of cylindrical waveguides with inhomogeneous filling addressing the case when permittivity of dielectric media is a function of the radial polar coordinate. Note that we consider only the waves decaying with respect to the distance from the waveguide external boundary, namely, the surface waves (imposing the appropriate conditions at infinity). Other types of waves are not considered.

2 Open inhomogeneous waveguide. Statement of the problem

Consider three-dimensional space $\mathbb{R}^3$ equipped with cylindrical coordinate system $O\rho\varphi z$ and filled with an isotropic source-free medium having the permittivity $\varepsilon = \varepsilon_0\varepsilon_c \equiv const$, where ε_0 is the permittivity vacuum. An open single-layer cylindrical metal-dielectric waveguide, the Goubau line (GL), with circular cross-section

$$\Sigma := \{(\rho,\varphi,z) : r_0 \leqslant \rho \leqslant r, 0 \leqslant \varphi \leqslant 2\pi\}$$

with a generatrix parallel to the Oz axis is placed in $\mathbb{R}^3$.

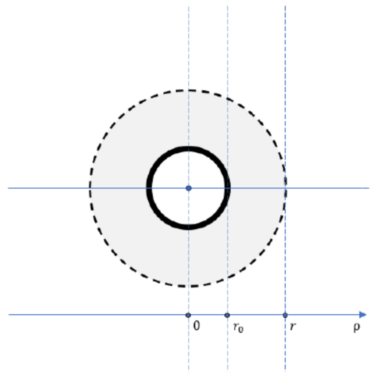

Figure 1. Cross-section of a waveguide Σ.

The cross-section of the waveguide by a plane perpendicular to its axis consists of a ring with internal and external radii r_0 and r. The circles $\rho = r_0$ and $\rho = r$ are the projections of the surfaces of a perfectly conducting infinitely thin screen and interface of the dielectrics, respectively.

The problem of the electromagnetic TE-polarized wave propagation in this open metal-dielectric waveguide consists in finding nontrivial solutions of the homogeneous system of Maxwell's equations in the form of traveling waves [16, **?**], i.e. with the dependence $e^{i\gamma z}$ on the coordinate z along which the structure is regular:

$$\begin{cases} \operatorname{rot}\mathbf{H} = -i\omega\varepsilon\mathbf{E}, \\ \operatorname{rot}\mathbf{E} = i\omega\mathbf{H}, \end{cases} \tag{1}$$

$$\mathbf{E} = \left(0, E_\varphi(\rho)\mathbf{e}_\varphi, 0\right)e^{i\gamma z},\ \mathbf{H} = \left(H_\rho(\rho)\mathbf{e}_\rho, 0, H_z(\rho)\mathbf{e}_z\right)e^{i\gamma z};$$

in addition, the following conditions must be satisfied: boundedness of the field energy in any finite volume of the waveguide, vanishing of the tangential electric field components on the surface of a perfect conductor

$$E_\varphi\big|_{\rho=r_0}=0; \tag{2}$$

continuity of the tangential components at the interface

$$[E_\varphi]\big|_{\rho=r}=0,\ [H_z]\big|_{\rho=r}=0, \tag{3}$$

and the radiation condition at infinity which will be formulated and discussed later.

We assume that permittivity in the entire space has the form $\varepsilon=\widetilde{\varepsilon}\varepsilon_0$, where

$$\widetilde{\varepsilon}=\begin{cases}\varepsilon_r(\rho), & r_0\leqslant\rho\leqslant r,\\ \varepsilon_c, & \rho>r,\end{cases} \tag{4}$$

where ε_c is a real positive constant, and that $\varepsilon_r(\rho)$ is a twice continuously differentiable function on the interval $[r_0,r]$, i.e. $\varepsilon(\rho)\in C^2[r_0,r]$.

Denoting $u(\rho):=E_\varphi(\rho)$, and setting $k_0^2:=\omega^2\mu_0\varepsilon_0$, we obtain

$$\left(\rho^{-1}(\rho u)'\right)'+(k_0^2\widetilde{\varepsilon}-\gamma^2)u=0, \tag{5}$$

where the derivative denotes differentiation with respect to ρ and $u(\rho,\gamma)$ is a real function.

For $\rho>r$, we have $\widetilde{\varepsilon}=\varepsilon_c$, then from (7) we obtain the Bessel equation

$$u''+\rho^{-1}u'-\rho^{-2}u-\kappa^2u=0, \tag{6}$$

where $\kappa^2=\gamma-k_0^2\varepsilon_c$. The solution of (8) has the form

$$u=\widetilde{C}I_1(\kappa\rho)+CK_1(\kappa\rho),\rho>r,$$

where C and $\widetilde{C}$ are constants. It is known [17] that $I_1(\rho)$ tends to infinity as $\rho\to+\infty$, and $K_1(\rho)$ tends to zero as $\rho\to+\infty$. Taking into account these properties and the condition at infinity, we obtain that $\widetilde{C}=0$ and

$$u=CK_1(\kappa\rho),\rho>r. \tag{7}$$

In the waveguide cladding $r_0\leq\rho\leq r$, we have $\widetilde{\varepsilon}=\varepsilon_r(r)$. Then, from (7) we obtain the following equation:

$$u''+\rho^{-1}u'-\rho^{-2}u+(k_0^2\varepsilon_r-\gamma^2)u=0. \tag{8}$$

The tangential electromagnetic field components are continuous at the interface between the media. From the continuity condition we obtain

$$[u]\big|_{\rho=r}=0,\ [u']\big|_{\rho=r}=0, \tag{9}$$

Finally, since the tangential components of the electric field vanish on the surface of a perfect conductor, we get the boundary condition for $u(\rho)$

$$u|_{\rho=r_0}=0, \tag{10}$$

Definition 1. Problem P_0: find $\gamma\in\mathbb{R}$ such that there exist nontrivial solutions u to differential equation (10) satisfying conditions (11) and (12).

3 Closed inhomogeneous waveguide. Formulation of the problem

A closed (shielded) two-layer cylindrical metal-dielectric waveguide

$$\Pi:=\{(\rho,\varphi,z):r_0\leqslant\rho<r,0\leqslant\varphi<2\pi\}\cup\\ \{(\rho,\varphi,z):r_0\leqslant\rho\leqslant R,0\leqslant\varphi<2\pi\}$$

with a generatrix parallel to the Oz axis and a circular cross-section is placed in $\mathbb{R}^3$.

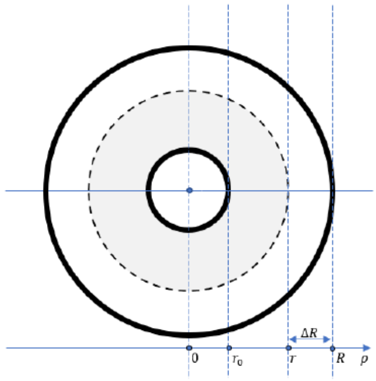

Figure 2. Cross-section of a waveguide Π.

The cross-section of the waveguide by a plane perpendicular to its axis consists of two rings with internal radii r_0 and r and and external radii r and R, respectively. The circles $\rho=r_0$ and $\rho=R$ are the projections of the surfaces of a perfectly conducting infinitely thin screen and $\rho=r$ of the interface of dielectrics. Denote by $\triangle R=R-r$ the thickness of the external layer.

The problem of the TE-polarized wave propagation in the waveguide consists in finding nontrivial solutions of the homogeneous system of Maxwell's equations in the form of a traveling wave having the dependence $e^{i\gamma z}$ on coordinate z:

$$\begin{cases}\operatorname{rot}\mathbf{H}=-i\omega\varepsilon\mathbf{E},\\ \operatorname{rot}\mathbf{E}=i\omega\mathbf{H},\end{cases} \tag{11}$$

$$\mathbf{E}=\left(0,E_\varphi(\rho)\mathbf{e}_\varphi,0\right)e^{i\gamma z},\ \mathbf{H}=\left(H_\rho(\rho)\mathbf{e}_\rho,0,H_z(\rho)\mathbf{e}_z\right)e^{i\gamma z};$$

the following conditions must be satisfied: boundedness of the field energy in any finite volume of the waveguide,

$$E_\varphi\big|_{\rho=r_0}=0,\ \text{and}\ E_\varphi\big|_{\rho=R}=0; \tag{12}$$

and the continuity of tangential components of fields at the interface

$$[E_\varphi]\big|_{\rho=r}=0,\ [H_z]\big|_{\rho=r}=0, \tag{13}$$

We assume that permittivity in the entire space has the form $\varepsilon=\widetilde{\varepsilon}\varepsilon_0$, where

$$\widetilde{\varepsilon}=\begin{cases}\varepsilon_r(\rho), & r_0\leqslant\rho\leqslant r,\\ \varepsilon_c, & r\leqslant\rho\leqslant R,\end{cases} \tag{14}$$

Denoting $v(\rho) := E_\varphi(\rho)$, we obtain

$$\left(\rho^{-1}(\rho v)'\right)' + (k_0^2\tilde{\varepsilon} - \gamma^2)v = 0, \qquad (15)$$

In the inner layer of the waveguide $r \leqslant \rho \leqslant R$, we have $\tilde{\varepsilon} = \varepsilon_c$, then from (19) we obtain the Bessel equation

$$v'' + \rho^{-1}v' - \rho^{-2}v - \kappa^2 v = 0, \qquad (16)$$

The solution of (20) has the form

$$v = \tilde{C}I_1(\kappa\rho) + CK_1(\kappa\rho), r \leqslant \rho \leqslant R,$$

where C and $\tilde{C}$ are constants. Taking into account condition (16) on the external boundary $\rho = R$, we get

$$v = \tilde{C}(I_1(\kappa R)K_1(\kappa\rho) - K_1(\kappa R)I_1(\kappa\rho)), r \leqslant \rho \leqslant R. \quad (17)$$

In the waveguide cladding $r_0 \leq \rho \leq r$ we have $\tilde{\varepsilon} = \varepsilon_r(r)$. Then, from (7) we obtain the following equation:

$$v'' + \rho^{-1}v' = \rho^{-2}v + (k_0^2\varepsilon_r - \gamma^2)v = 0. \qquad (18)$$

From the condition of continuity of the tangential components, we obtain

$$[v]|_{\rho=r} = 0, \;\; [v']\big|_{\rho=r} = 0, \qquad (19)$$

Definition 2. Problem P_C: find $\gamma \in \mathbb{R}$ such that there exist nontrivial solutions v to differential equation (22) satisfying conditions (15) and (23).

4 Numerical method

Consider the Cauchy problem for the equation

$$w' = -\rho^{-1}w' + \rho^{-2}w - (k_0^2\varepsilon_r - \gamma^2)v = 0, \qquad (20)$$

with initial conditions

$$w(r_0) = 0, w'(r_0) := A \qquad (21)$$

where A is a given constant (field amplitude), different from zero.

Remark 1. Note that for linear media, the propagation constants (eigenvalues) do not depend on the field amplitude at one of the waveguide boundaries.

We assume that a solution to the Cauchy problem exists, is unique, and is defined on the whole interval $[r_0, r]$ for given values of r_0, r and continuously depends on the parameter $\gamma > \omega\sqrt{\varepsilon_c}$.

From the conjugation condition on the second boundary r, we obtain the equation

$$\Phi(\gamma) \equiv \phi_1(\gamma)w(r-0) + \phi_2(\gamma)w'(r-0), \qquad (22)$$

where $\phi(\gamma) = \kappa K_0(\kappa r) + r^{-1}K_1(\kappa r), \phi_2(\gamma) = K_1(\kappa r)$, for Problem P_O; and

$$\begin{aligned}\phi(\gamma) = \kappa(K_0(\kappa r)I_1(\kappa R) + I_0(\kappa r)K_1(\kappa R)) + \\ + r^{-1}(K_0(\kappa r)I_1(\kappa R) - I_0(\kappa r)K_1(\kappa R)), \\ \phi_2(\gamma) = (K_1(\kappa r)I_1(\kappa R) - I_1(\kappa r)K_1(\kappa R)),\end{aligned}$$

for Problem P_C.

From formula (26) it follows that the value of $\Phi(\gamma)$ is expressed only through the values of the solution to the Cauchy problem $w(r)$ and $w'(r)$. Assume that $\gamma = \tilde{\gamma}$ is such that $\Phi(\gamma) = 0$; then it is clear that $\tilde{\gamma}$ is a solution (a propagation constant).

Statement 1. *Let the segment* $[\underline{\gamma}, \bar{\gamma}]$ *be such that* $\Phi(\underline{\gamma})\Phi(\bar{\gamma}) < 0$. *Then, there exists at least one propagation constant (one eigenvalue)* $[\tilde{\gamma} \in \underline{\gamma}, \bar{\gamma}]$.

The set of solutions to the equation $\Phi(\gamma) = 0$ determines the spectrum of eigenwaves of the propagating TE-polarized waves. We will call $\Phi(\gamma) = 0$ the dispersion equation of the problem. By solving numerically the equation $\Phi(\gamma) = 0$ (for different values of frequency), it is possible to construct graphs of the dependence (dispersion curves) of eigenvalues γ on frequency f. Note that requency f is related to cyclic frequency ω by the relation $f = 2pi\omega$.

Figures 3 and 4 show the calculated solutions to the dispersion equation for Problems P_0 and P_C, respectively.The gray lines mark the domain defined by the condition $k_0^2\varepsilon < \gamma^2 < k_0^2 \max\limits_{[r_0,r]}(\varepsilon_r)$, where the problems under consideration have solutions. The values of parameters used in calculations are indicated in the captions to the graphs. Figures 3 and 4 show that the spectra of single-layer closed and open waveguides (with the same parameters), as expected, are different. What is more, for the chosen frequency $\omega = 5.5$, we have different number of eigenvalues.

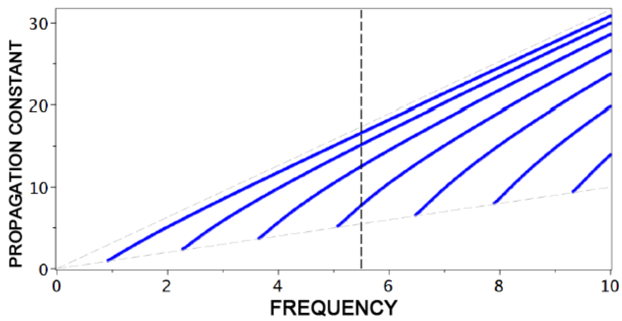

Figure 3. Fig. 3: Problem P_0: dispersion curves. Parameter values: $A = 1, r_0 = 0.25, r = 1.0, \varepsilon_c = 1, \varepsilon_r = 9 + \rho$.

Let us call how the number and magnitude of the eigenvalues of Problem P_C change with the increasing thickness of the outer layer $\triangle R$, in comparison with eigenvalues of Problem P_0. We see (Fig. 5) that already when the outer layer

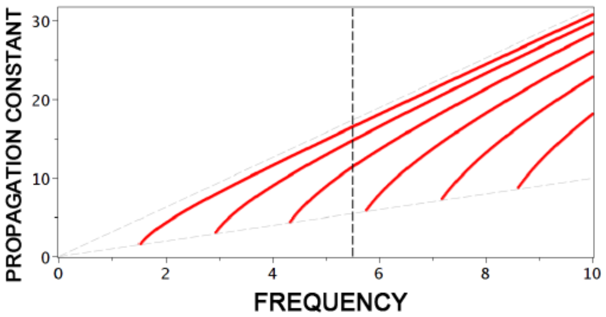

Figure 4. Fig. 4: Problem P_C: dispersion curves. Parameter values: $A = 1, r_0 = 0.25, r = 1.0, \triangle R = 0, \varepsilon_c = 1, \varepsilon_r = 9 + \rho$.

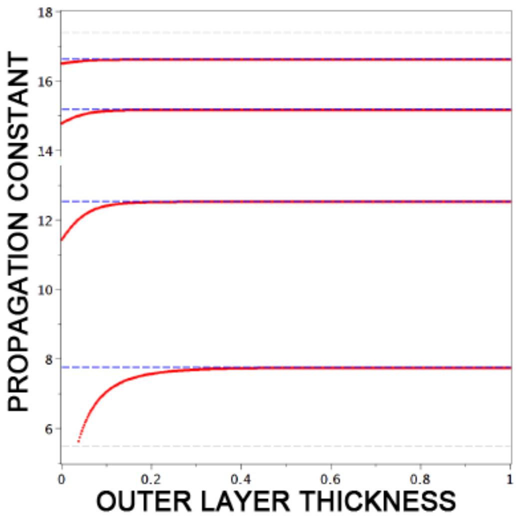

Figure 5. Fig. 5: Comparison of the spectra of Problems P_O (blue lines) and P_C (red lines). Parameter values: $A = 1, r_0 = 0.25, r = 1.0, \varepsilon_c = 1, \varepsilon_r = 9 + \rho$.

thickness $\triangle R > 0.5$, the number and magnitude of the eigenvalues of Problems P_O (blue curves) and P_C (red curves) coincide.

We can draw the following important conclusion: the spectra of Problems P_O and P_C coincide for sufficient thick external layers.

5 Conclusion

A comparison of the properties of the surface TE-polarized electromagnetic waves in GL and a two-layer shielded dielectric waveguide filled with an inhomogeneous medium is performed as a result of series of calculations completed using a specially constructed numerical method. The obtained numerical results of the comparison confirm several important properties of the TE-wave spectra which have not been reported in earlier studies.

6 Acknowledgements

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