Excitation of a Layered Sphere by Multiple Point-Generated Primary Fields

Andreas Kalogeropoulos and Nikolaos L. Tsitsas

School of Informatics, Aristotle University of Thessaloniki, Thessaloniki, Greece

akaloger@csd.auth.gr  ntsitsas@csd.auth.gr
Outline

- Excitation by multiple sources: background and motivation
- Mathematical formulation
- Excitation operators and overall fields
- Derivation of the exact solution of the direct problem
- Low-frequency approximations
- Numerical results
- Conclusions
Background

- **T-Matrix method**: A method for the solution of the direct scattering problem, that reduces the problem of identifying the exact form of the scattered fields into solving $2 \times 2$ linear systems.

- It was first introduced by *Peter Waterman* in 1969 and since, it has been established as an effective approach for scattering problems involving stratified media or periodic structures.

- **Excitation by multiple sources**: In many real-world problems, the overall radiation is a result of excitation by many different sources, e.g. electromagnetic activity of the brain, isotropic radiators, wind profiling SODARs, etc.

- Inverse schemes exploiting *a priori* assumptions for the nature of these sources are often used, e.g. beamforming techniques using microphone arrays (aero-acoustics) or dipole arrays (electromagnetics).
Objectives

- A layered medium excited by multiple point-generated fields: useful and realistic model for a variety of applications

- Present research objectives:
  - Formulation of the problem based on the T-Matrix approach
  - Derivation for the exact solution of the direct problem
  - Low-frequency approximations for the overall far-field and the overall scattering cross section
  - Preliminary numerical implementations
Motivating applications

- Activity of the human body (e.g. brain, heart)
  - “the field generated by the heart may be regarded as not significantly different from that of a dipole at the center of a homogeneous spherical conductor”
    - Wilson, Bayley, Circulation, 1950
- Biomedical (biotelemetry, cancer treatment, hyperthermia)
  - dipole implantations in the head
- Electroencephalography and medical imaging
  - source inside the brain determined by scalp measurements
    - Dassios, Fokas, Inverse Problems, 2009
  - brain imaging:
    - modeling by point-dipoles inside spherical or ellipsoidal media
      - Dassios, Lect. Notes Math., 2009
      - Ammari, Introduction to Mathematics of Emerging Biomedical Imaging, 2008
Motivating applications

- Method of point sources and partial waves
  - Approximation by multiple point sources
    - Potthast, *Point Sources and Multipoles in Inverse Scattering Theory*, 2001

- Antenna design
  - Microstrip antennas, dipole arrays, RFID antennas

- Meteorology
  - Wind profiling via SODAR
Mathematical formulation

\( N \) point sources located at the sphere’s exterior, generate spherical acoustic fields

\[
\mathbf{u}^{pr}(\mathbf{r}; \mathbf{r}_j) = A_j \frac{\exp(ik_0|\mathbf{r} - \mathbf{r}_j|)}{|\mathbf{r} - \mathbf{r}_j|}, \; \mathbf{r} \neq \mathbf{r}_j
\]

The layers of the sphere, are composed of materials with wavenumbers \( k_p \) and mass densities \( \rho_p \) (\( p=1,\ldots,P-1 \)).

Core \( V_P \) can be soft, hard or penetrable.
Mathematical formulation

- The individual and overall time-harmonic fields satisfy the scalar Helmholtz equations in the layers $V_p$

\[ \nabla^2 u^p(\mathbf{r}; \cdot) + k_p^2 u^p(\mathbf{r}; \cdot) = 0, \]

- the transmission boundary conditions on $S_p$ ($p=1, \ldots, P-1$)

\[ u^{p-1}(\mathbf{r}; \cdot) = u^p(\mathbf{r}; \cdot), \quad r = a_p \]

\[ \frac{1}{\rho_{p-1}} \frac{\partial u^{p-1}(\mathbf{r}; \cdot)}{\partial n} = \frac{1}{\rho_p} \frac{\partial u^p(\mathbf{r}; \cdot)}{\partial n}, \quad r = a_p \]

- and the Dirichlet or the Neumann boundary conditions on the core

\[ u^{p-1}(\mathbf{r}; \cdot) = 0, \quad r = a_p \]

\[ \frac{\partial u^{p-1}(\mathbf{r}; \cdot)}{\partial n} = 0, \quad r = a_p \]

or the transmission boundary conditions for a penetrable core

- The external field satisfies also the Sommerfeld radiation condition.
The term *individual fields* is referred to the fields induced due to a single point source exciting the scatterer. Utilizing the free-space Green's function, the individual primary and secondary fields are given, respectively, by:

\[
\begin{align*}
    u_0^{pr}(\mathbf{r}; \mathbf{r}_j) &= 4\pi ik_0A_j \left\{ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^m Y_n^{-m}(\hat{\mathbf{r}}_j)Y_n^m(\hat{\mathbf{r}}) \\
    & \quad + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^m Y_n^m(\hat{\mathbf{r}}_j)Y_n^{-m}(\hat{\mathbf{r}}) \right. \\
    & \quad \left. j_n(k_0r)h_n(k_0r_j), \quad r > r_j \\
    & \quad j_n(k_0r)h_n(k_0r_j), \quad r < r_j, \right.
\end{align*}
\]

\[
\begin{align*}
    u^P(\mathbf{r}; \mathbf{r}_j) &= 4\pi ik_0A_j \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^m Y_n^{-m}(\hat{\mathbf{r}}_j)Y_n^m(\hat{\mathbf{r}}) \\
    & \quad h_n(k_0r_j)(a_{j,n}^P j_n(k_pr) + b_{j,n}^P h_n(k_pr)),
\end{align*}
\]
The overall field of layer \( V_p \) is the superposition of all individual fields induced in \( V_p \) by the external point sources. In the exterior \( V_0 \) of the sphere the external field, is the superposition of all primary and secondary fields.

We define the following excitation operators which simplify the solution of the direct problem, by means of the T-Matrix approach:

\[
\mathcal{J}_{n,m}(x) = \sum_{j=1}^{N} A_j Y_n^{-m}(\hat{r}_j) j_n(k_0 r_j) x_j,
\]

\[
\mathcal{H}^{1}_{n,m}(x) = \sum_{j=1}^{N} A_j Y_n^m(\hat{r}_j) h_n(k_0 r_j) x_j,
\]

\[
\mathcal{H}^{2}_{n,m}(x) = \sum_{j=1}^{N} A_j Y_n^{-m}(\hat{r}_j) h_n(k_0 r_j) x_j,
\]

\[
x = (x_1, \ldots, x_N)
\]
With the aid of the excitation operators, the overall primary field in $V_0$ is given by the following formula:

$$u^{pr}(r; r_1, \ldots, r_N) = 4\pi ik_0 \left\{ \begin{array}{l} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^m Y_n^m(\hat{r}) h_n(k_0r) \mathcal{J}_{n,m}(q), \quad r > \max[r_j] \\ \vdots \\ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^m Y_n^{-m}(\hat{r}) j_n(k_0r) \mathcal{H}_{n,m}^1(q), \quad r < \min[r_j], \end{array} \right.$$

$q = (1, 1, \ldots, 1)$

The secondary field in layer $V_p$ is given by

$$u^p(r; r_1, \ldots, r_N) = 4\pi ik_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^m Y_n^m(\hat{r}) \left( \mathcal{A}_{n,m}^p j_n(k_pr) + \mathcal{B}_{n,m}^p h_n(k_pr) \right).$$

$$a_n^p = (a_{1,n}^p, \ldots, a_{N,n}^p)$$
$$b_n^p = (b_{1,n}^p, \ldots, b_{N,n}^p)$$

$$\mathcal{A}_{n,m}^p = \mathcal{H}_{n,m}^2(a_n^p) \quad \text{and} \quad \mathcal{B}_{n,m}^p = \mathcal{H}_{n,m}^2(b_n^p)$$
The transition matrix from layer $V_{p-1}$ to layer $V_p$ is given by

$$T^n_p = -ik_p^2a_p^2 \begin{bmatrix}
    h'_n(x_p)j_n(y_p) - w_j h_n(x_p)j'_n(y_p) & h'_n(x_p)h_n(y_p) - w_j h_n(x_p)h'_n(y_p) \\
    w_j j_n(x_p)j'_n(y_p) - j'_n(x_p)j_n(y_p) & w_j j_n(x_p)h'_n(y_p) - j'_n(x_p)h_n(y_p)
\end{bmatrix}$$

Notation for the transition matrix from layer $V_p$ to layer $V_s$ and boundary transition vector:

$$x_p = k_p a_p, \; y_p = k_{p-1} a_p \; \text{and} \; w_p = (k_{p-1} \rho_p)/(k_p \rho_{p-1})$$

$$\Psi_n(x) = (T^{(0\rightarrow P-1)}_n)^T \begin{bmatrix} f_n(x) \\ g_n(x) \end{bmatrix}$$

$$T^{(0\rightarrow p)}_n = T^n_p \cdot T^{p-1}_n \cdot \ldots \cdot T^1_n$$

$$f_n(x) = \begin{cases}
    j_n(x), & \text{soft core} \\
    j'_n(x), & \text{hard core}
\end{cases}$$

$$g_n(x) = \begin{cases}
    h_n(x), & \text{soft core} \\
    h'_n(x), & \text{hard core}
\end{cases}$$
A straightforward implementation of the conditions on the boundaries of layers $V_1 \ldots V_{P-1}$ yields:

$$
\begin{bmatrix}
A_{n,m}^{P-1} \\
B_{n,m}^{P-1}
\end{bmatrix} = T_{n}^{(0\rightarrow P-1)}
\begin{bmatrix}
\mathcal{H}_{n,m}^1(q) \\
\mathcal{B}_{n,m}^0
\end{bmatrix}
$$

For a soft or hard core the coefficient of the overall secondary field is given by:

$$
B_{n,m}^0 = -\frac{\Psi_n^1(k_{P-1}a_P) \mathcal{H}_{n,m}^1(q)}{\Psi_n^2(k_{P-1}a_P)}
$$

For a penetrable core we obtain:

$$
B_{n,m}^0 = -\frac{T_{n}^{(0\rightarrow P)} \mathcal{H}_{n,m}^1(q)}{T_{22,n}^{(0\rightarrow P)}}
$$
The coefficients of the individual external secondary fields, for a soft/hard or penetrable core, are given, respectively, by the formulas:

\[
b_{j,n}^0 = - \frac{\Psi_1^{1}(k_{P-1}a_P) \mathcal{H}_{n,m}^1(h_j)}{\Psi_2^{2}(k_{P-1}a_P)}
\]

\[
b_{j,n}^0 = - \frac{T_{21,n}^{(0\rightarrow P)} \mathcal{H}_{n,m}^1(h_j)}{T_{22,n}^{(0\rightarrow P)}}
\]

where

\[
h_j = \frac{e_j}{A_j Y_{n-m}^{-m}(\hat{r}_j) h_n(k_0 r_j)}
\]

The overall far-field and the overall scattering cross section, are given by

\[
g(\hat{r}) = 4\pi i \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-1)^m (-i)^n Y_n^m(\hat{r}) B_{n,m}^0
\]

\[
\sigma = \frac{4\pi}{k_0^2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1) \frac{(n-m)!}{(n+m)!} |B_{n,m}^0|^2
\]
Low-Frequency Approximations

Under the low-frequency assumption $k_0a_1 \to 0$ the overall far-field and the overall scattering cross section take the forms:

$$g(\mathbf{r}) = \kappa NS_1^0 + \kappa^2 \left(N \rho \eta (S_1^0)^2 + S_2^0 \sum_{j=1}^{N} \tau_j f_j(\theta, \phi) \right) + \kappa^3 \left(N \beta(\rho, \xi, \eta)(S_1^0)^3 - S_2^0 \sum_{j=1}^{N} f_j(\theta, \phi) + S_3^0 \sum_{j=1}^{N} \tau_j^2 F_j(\theta, \phi) \right)$$

$$F_j(\theta, \phi) = \sin^2 \theta \sin^2 \theta_j \cos(\phi_j - \phi) + \sin^2 \theta \sin^2 \theta_j \cos(2(\phi_j - \phi)) + \left(\cos^2 \theta_j - \frac{1}{3}\right)(3 \cos^2 \theta - 1)$$

$$f_j(\theta, \phi) = \cos \theta_j \cos \theta + \sin \theta_j \sin \theta \cos(\phi_j - \phi)$$

$$\sigma = 4\pi a_1^2 \left[ N^2(S_1^0)^2 + (k_0a_1)^2 \left(N^2(S_1^0)^4 \delta(\rho, \eta, \xi) + \frac{(S_2^0)^2}{3} \left(\sum_{j=1}^{N} \tau_j^2 + 2 \sum_{j=1}^{N-1} \sum_{\nu=j+1}^{N} \tau_j \tau_\nu f_j(\theta_\nu, \phi_\nu) \right) \right) \right]$$

$$S_1^0 = \frac{1}{\rho - 1 - \rho \xi}, \quad S_2^0 = \frac{\xi^3(1 - \rho) + 2 + \rho}{\xi^3(1 + 2\rho) + 2 + 2\rho}, \quad S_3^0 = -\frac{2\xi^5(1 - \rho) + 3 + 2\rho}{2\xi^5(2 + 3\rho) + 3 - 3\rho},$$

$$\beta(\rho, \eta, \xi) = \frac{(\rho\eta)^2(2\xi + 1) - \rho \eta^2}{3\xi}, \quad \delta(\rho, \eta, \xi) = (\rho\eta)^2 \left(1 - \frac{\rho(2\xi + 1) - 1}{3\xi \rho} \right)$$
Numerical results

Low-frequency approximation vs. exact scattering cross section for a 2-layered sphere with a soft core with relative mass density index $\rho=1.5$ and refractive index $\eta=1.75$
Numerical results

Low-frequency approximation vs. exact scattering cross section for a 2-layered sphere with a soft core with relative mass density index $\rho=2.5$ and refractive index $\eta=2.25$
Conclusions

- Acoustic excitation of a layered sphere by $N$ external point sources
- Motivating applications
- Mathematical formulation of the direct scattering problem and suitable definitions of fields, far-field patterns and scattering cross sections
- Overall fields, excitation operators and T-Matrix formulation
- Exact solution of the direct scattering problem
- Low-frequency approximations
- Numerical results