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# On the resonant interaction of relativistic electrons with oblique whistler-mode waves.

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## Abstract

In the present study, we consider the problem of resonant interaction of the relativistic electrons with oblique monochromatic whistler-mode waves in the Earth's magnetosphere. The Hamiltonian theory of such an interaction has been developed, taking into account the peculiarities of relativistic case, e.g., the possibility for a resonance momentum to become zero on the particle trajectory. Using the approach of geometric optics we have calculated the field distribution of whistler-mode waves generated in the ionosphere by VLF transmitter or a lightning discharge. The motion of trapped electron in such a field have been investigated.

## 1. Introduction

The Earth's radiation belts were discovered experimentally in 1958 by Explorer mission under the guidance of Van Allen [1] and, independently, by Sputnik mission lead by Vernov [2]. Radiation belts represent two toroidal regions in the Earth's magnetosphere, where the energetic particles are trapped due to the mirror configuration of the Earth's magnetic field. The outer radiation belt located at 3.5-10 Earth radii (RE) is highly dynamical. It consists of energetic electrons of the energies up to several MeV, that is, relativistic energies, and its structure is strongly dependent on geomagnetic conditions. Since the discovery of the radiation belts the mechanisms of particle energization and precipitation in the outer radiation belt have become one of the most important problems of the physics of the Earth's magnetosphere, in particular, because of the threat to the electronic equipment on the satellites posed by highly energetic electrons [3].

Experimental data [4] confirms that the local acceleration mechanisms play an important role in the dynamics of electrons of the outer radiation belt. One possible mechanism is the local acceleration due to the resonant interaction with the waves in whistler frequency range (3 - 30 kHz in the inner magnetosphere). The theory of such an interaction of non-relativistic electrons with the waves propagating at arbitrary angle with respect to the Earth's magnetic field is very well developed [5]. Studying the interaction of whistler-mode waves propagating along magnetic field lines with relativistic electrons, Omura et al. [6, 7] have found very efficient mechanism of acceleration, the so called relativistic turning acceleration (RTA). Still, there is no general Hamiltonian theory for the resonant interaction of relativistic electrons with oblique whistler-mode waves.

In this report, we present the Hamiltonian theory for the resonant interaction of relativistic electrons with oblique whistler-mode waves. With the theory developed we have performed numerical simulations of such an interaction for the monochromatic whistler-mode waves propagating in the magnetosphere from the ionosphere, where they can be generated by, e.g., emissions of VLF transmitters or lightning discharges.

## 2. Layout

In Section 3, we present the main equations of the theory and analyze the structure of the resonance conditions. The wave-field assignment for the numerical simulations, and the results of numerical simulations are discussed in Section 4.

## 3. Particle dynamics and resonance condition

The main idea of the Hamiltonian approach to the description of the resonant interaction is to use the canonical transformations to consistently distinguish different resonances and to obtain the resonance Hamiltonian. The exact Hamiltonian for a relativistic particle in the external electromagnetic field is:

$$H = mc^2\sqrt{1+\frac{1}{m^2c^2}\left(\mathbf{P}+\frac{e}{c}(\boldsymbol{A}+\mathbf{A_0})\right)} \qquad (1)$$

Here, $\mathbf{A}(\mathbf{r},\ t)$ is the vector-potential of the wave field, $\mathbf{A}_0(\mathbf{r})$ is the vector-potential of the Earth's magnetic field, $\mathbf{P}$ is the canonical momentum. Assuming the wave amplitude to be small enough and the gyroradius much smaller than the inhomogeneity scale, one obtains the following Hamiltonian:

$$H = mc^2\gamma_0\left(p_{||}, s, \mu\right) - \frac{eA_w}{\gamma_0}\sum_n V_n \sin\xi_n, \qquad (2)$$

where $\gamma_0$ is the Lorentz factor

$$\gamma_0 = \sqrt{1+\frac{2\mu\omega_{\rm c}}{mc^2}+\frac{p_{||}^2}{m^2c^2}}$$

and $p_{||}$ is the particle momentum along field line, $s$ is the corresponding coordinate, $\mu$ is the particle magnetic moment, $\omega_c$ is electron cyclotron frequency (with respect to the Earth's magnetic field in the current particle position), $A_w$ is the wave amplitude, $V_n$ are the coefficients of partial interaction amplitudes, the same as for the non-relativistic case [5], and $\xi_n$ is the phase, combining the particle gyrophase $\varphi$ and wave phase:

$$\xi_n = \int k_{||} ds + n\varphi - \omega t$$

Here, $k_{||}$ is the component of the wave vector along field line, $n$ is the resonance number, $\omega$ is the wave frequency. The resonance condition is the condition of slow variation of the phase; it leads to the following equation:

$$p_{||}^{(n)} = m\frac{\omega\gamma_0(p_{||},\ s,\ \mu) - n\omega_c(s)}{k_{||}(s)} \tag{3}$$

In non-relativistic case, such equation defines the resonance momentum as the function of coordinate $s$. As follows from (3), the relativistic resonance condition defines a surface in $(p_{||},\ s,\ \mu)$-coordinates. But for the monochromatic waves we still can select the resonance curves $p_{||}(s)$ on this surface with the help of the integral of motion

$$h_n = nH - \mu\omega \tag{4}$$

Combining (3) and (4) one obtains the following equation for the resonance Lorentz factor:

$$F[\gamma_0] = {\gamma_0}^2 - 2n\frac{\omega_c}{\omega}\gamma_0 + \left(2\frac{\omega_c}{\omega}\frac{h_n}{mc^2} - 1 - \left(\frac{n\omega_c}{ck_{||}}\right)^2\right)\frac{1}{1-\omega^2/{k_{||}}^2c^2} = 0$$

This is the quadratic equation, and $F[\gamma_0]$ is the quadratic function. So one can easily see the structure of the sections of the resonance surface by (4). Figure 1 shows $F[\gamma_0]$ for all the parameters fixed. The minimum of the parabola is always at $\gamma_0 = n\frac{\omega_c}{\omega}$, and only the ordinate of the minimum depends on $h_n$. The physical region of the solutions is, of course, $\gamma_0 > 1$. Hence, when we start from $h_n \ll 0$ we always have only one (right, see Figure 1) root of the resonance equation (3)-(4). This root corresponds to the positive resonance momentum (3,5) (assuming $k_{||}$ to be positive). Increasing $h_n$, we move the parabola upwards, so at some $h_n$ the left root comes into the physical region. This root corresponds to the negative resonance momentum, such a resonant particle moves towards the wave. From now, we have two roots of the resonance equation, corresponding to the different signs of resonance momentum:

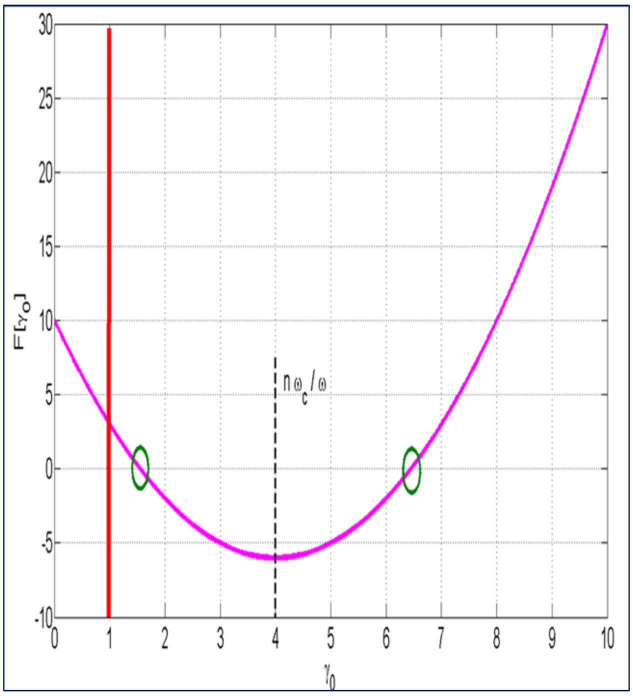

**Figure 1.** The solutions of the resonance equation.

$$p_{||}^{(n)} = \pm\frac{m\omega}{k_{||}}\sqrt{\left(\frac{n\omega_c}{\omega}\right)^2 - \left(2\frac{\omega_c}{\omega}\frac{h_n}{mc^2} - 1 - \left(\frac{n\omega_c}{ck_{||}}\right)^2\right)\frac{1}{1-\omega^2/{k_{||}}^2c^2}} \tag{5}$$

When the ordinate of the minimum reaches zero, the two roots merge, and there is only one resonance momentum which is equal to zero. It corresponds to the turning of the resonant particle.

One can show that the particle dynamics in the given resonance is defined by two parameters: the inhomogeneity parameter $\alpha \sim \partial\omega_c/\partial s$ and effective amplitude $\beta$:

$$\beta = \frac{{k_{||}}^2 m}{{\gamma_0}^2} eA_w V_n\left(1 - \frac{\omega^2}{{k_{||}}^2c^2}\right) \tag{6}$$

The important feature of oblique wave propagation here is that the effective amplitude depends on $V_n$ which is proportional to the Bessel functions $J_n(\rho)$ of dimensionless gyroradius $\rho = k_\perp\sqrt{2\mu/m\,\omega_c}$ .When $\beta > \alpha$ there are two types of resonant particles: trapped particles which momentum follows the resonance momentum, and untrapped particles which come into the resonance in a single moment of time.

## 4. Numerical simulations

We assume that the waves propagate in the magnetosphere from the ionosphere. Such waves can be generated by the emissions of the VLF transmitters or lightning discharges. To reconstruct the wave field in the magnetosphere along the field line we use the geometric optics [8]. The wave frequency was taken 5 kHz. The results are presented in Figure 2.

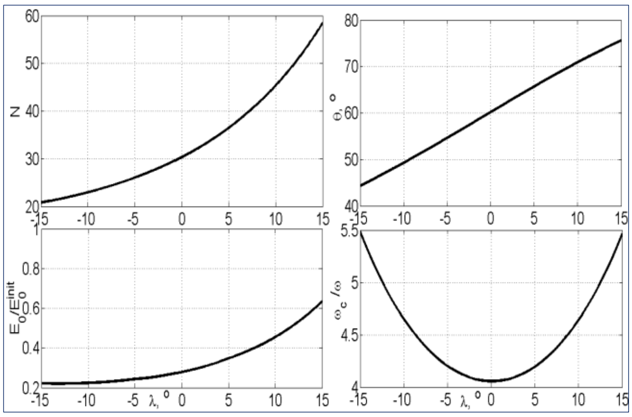


**Figure 2.** The wave parameters on the particle trajectory (L shell = 3.5) as a functions of geomagnetic latitude. *N* is the refractive index, $\theta$ is the wave normal angle.

Having calculated the wave parameters along the particle trajectory, we can find the resonance momenta for the particles with different $h_n$ in different resonances. The results for the first cyclotron resonance are presented in Figure 3.

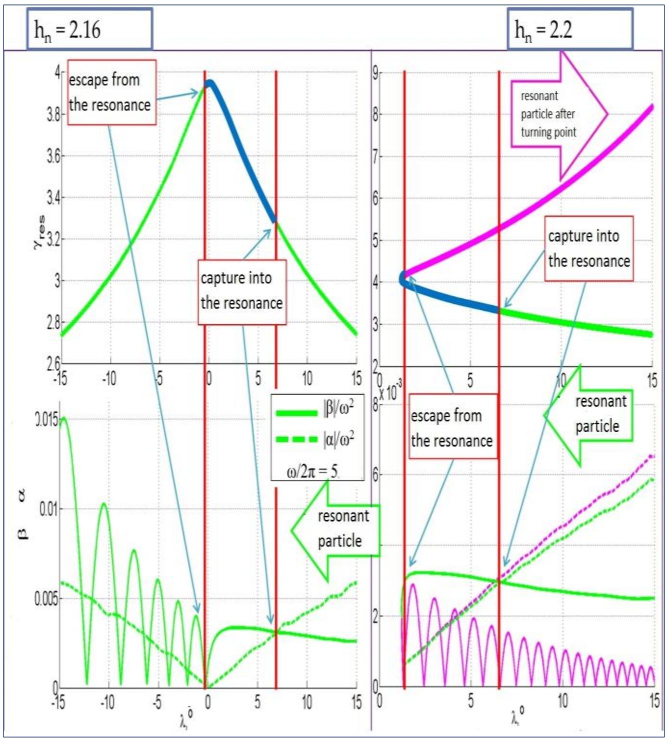


**Figure 3.** The particle resonance parameters as functions of geomagnetic latitude (L = 3.5). Upper panels: Lorentz factor, which is dimensionless energy. Lower panels: inhomogeneity parameter $\alpha$ and effective amplitude $\beta$. Green color corresponds to the negative solution for the resonance momentum (particle moves towards the wave), magenta color corresponds to the positive solution (5). When $\beta < \alpha$, trapped particle escapes from the resonance.

Figure 3 presents the following situation. Initially, there is a particle moving from positive latitudes to the equator (and towards the wave). When the effective amplitude becomes larger than inhomogeneity parameter, the particle can be trapped into the resonance. Then, the particle energy variation follows the resonance Lorentz factor. One can see that the energy increases towards the equator. Let us discuss the left panels of Figure 3. They correspond to the case when there is no resonance turning point on the trapped particle trajectory. The inhomogeneity parameter is almost symmetric with respect to the equator. But the effective amplitude is not: after the equator is starts oscillating. Oscillations of the effective amplitude lead to the particle escape from the resonance, preventing the energy exchange between the particle and the wave. But after crossing the equator such exchange would lead to particle deceleration, as can be seen from upper left panel of Figure 3. So such an asymmetry of the effective amplitude with respect to equator results in efficient energization of the trapped particles.
The opposite holds for the case, when there is a resonance turning point on the particle trajectory (see right panels of Figure 3). Now, the effective amplitude is asymmetric with respect to the resonance turning point, so the trapped particle escapes the resonance after the turning point, and is not being energized further. It can be shown that such an asymmetry arises for the relativistic effective amplitude due to the obliquity of wave propagation. The effective amplitude (6) depends on the Bessel functions of dimensionless gyroradius, and this dependency gives rise to the oscillations of the effective amplitude.

## 5. Conclusion

We investigated the structure of the resonant surface for relativistic resonant interaction of electrons with oblique whistler-mode wave.
There is an asymmetry of the effective amplitude with respect to equator and resonance turning point.
This asymmetry leads to efficient trapped particle energization when there is no resonance turning point, but reduces the efficiency of relativistic turning acceleration process.

## 6. Acknowledgements


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