Fresnel Reflection and Transmission Coefficients for Complex Media

Kurt E. Oughstun[(1)], and Christopher L. Palombini[(2)]
(1) EBE Department, University of Vermont, Burlington, VT, https://www.uvm.edu/cems/ebe
(2) Applied Physical Sciences Corp., Lexington, MA 02421-5411

1. Introduction

Fresnel reflection and transmission coefficients are well known for the two idealized cases when the two media are either perfect dielectrics [1] or one is a dielectric and the other is a perfect electric conductor (PEC) [2]. However, such is not the case when either one or both media have loss. In a recent paper, Besieris [3] showed that Canning's criticism [4] of the different approaches to the solution of this problem (see Roy [5]) was unnecessary as both are correct in describing the reflection and transmission coefficients when one medium was lossless and the other has loss. The approach taken by Canning [4], as reformulated by Besieris [3], is generalized here to the case when both media have loss, a situation that occurs in layered complex, lossy materials.

2. Formulation

Each temporally dispersive material is assumed to be homogeneous, isotropic, and locally linear, as described by the frequency-domain constitutive relations $\tilde{D}_j(r,\omega) = \epsilon_j(\omega)\tilde{E}_j(r,\omega)$, $\tilde{B}_j(r,\omega) = \mu_j(\omega)\tilde{H}_j(r,\omega)$, $\tilde{J}_{cj}(r,\omega) = \sigma_j(\omega)\tilde{E}_j(r,\omega)$, with complex-valued dielectric permittivity $\epsilon_j(\omega) = \epsilon'_j(\omega) + i\epsilon''_j(\omega)$, magnetic permeability $\mu_j(\omega) = \mu'_j(\omega) + i\mu''_j(\omega)$, and conductivity $\sigma_j(\omega) = \sigma'_j(\omega) + i\sigma''_j(\omega)$, $j = 1,2$. The *complex dielectric permittivity* of each material is then given by

$$\epsilon_{cj}(\omega) = \epsilon_j(\omega) + i\,\frac{\sigma_j(\omega)}{\omega}, \tag{1}$$

with *refractive index* $n_j(\omega) = \left(\mu_j(\omega)\epsilon_{cj}(\omega)/\mu_0\,\epsilon_0\right)^{1/2}$ having real $n'_j(\omega) = \Re\{n_j(\omega)\}$ and imaginary $n''_j(\omega) = \Im\{n_j(\omega)\}$ parts. The analysis is based on the frequency domain form of the tangential boundary conditions

$$\hat{n}\times\left(\tilde{E}_2(r,\omega) - \tilde{E}_1(r,\omega)\right) = \hat{n}\times\left(\tilde{H}_2(r,\omega) - \tilde{H}_1(r,\omega)\right) = 0\,, \tag{2}$$

for all $r \in S$, where $\hat{n}$ is the unit normal to the interface S, directed from medium 1 into medium 2. For convenience, S is situated in the xy-plane at $z = 0$ with unit normal $\hat{n} = \hat{1}_z$ directed from medium 1, where the incident and reflected fields reside, into medium 2, where the transmitted field resides. With the origin O fixed at a point on S so that $\hat{1}_z \cdot r = 0$ for all $r \in S$, the position vector $r \in S$ may then be expressed as $r = -\hat{1}_z\times(\hat{1}_z\times r)$.

3. Complex Fresnel Coefficients for *TE*-Polarization

For a linearly polarized time-harmonic electromagnetic plane wave incident upon the interface S in medium 1 with electric field vector $\hat{E}_i(r,\omega) = \hat{1}_y E_0 exp\left(i\tilde{k}_i(\omega)\cdot r\right)$, the incident wave vector is given by $\tilde{k}_i(\omega) = \vec{\beta}_i(\omega) + i\vec{\alpha}_i(\omega)$ with

$$\vec{\beta}_i(\omega) = \beta_1(\omega)\left[\hat{1}_x \sin\Theta_i + \hat{1}_z\cos\Theta_i\right] \quad \& \quad \vec{\alpha}_i(\omega) = \hat{1}_z\alpha_1(\omega)\,, \tag{3}$$

where $\beta_1(\omega) = (\omega/c)n'_1(\omega)$, $\alpha_1(\omega) = (\omega/c)n''_1(\omega)$, and Θ_i is the angle of incidence from the normal to the interface S. The reflected plane wave electric field vector in medium 1 is then given by $\hat{E}_r(r,\omega) = \hat{1}_y\Gamma_{TE}(\omega)E_0 exp\left(i\tilde{k}_r(\omega)\cdot r\right)$ with reflected wave vector $\tilde{k}_r(\omega) = \vec{\beta}_r(\omega) + i\vec{\alpha}_r(\omega)$ with

$$\vec{\beta}_r(\omega) = \beta_1(\omega)\left[\hat{1}_x \sin\Theta_r - \hat{1}_z\cos\Theta_r\right] \quad \& \quad \vec{\alpha}_r(\omega) = -\hat{1}_z\alpha_1(\omega)\,, \tag{4}$$

where Θ_r is the angle of reflection, and where $\Gamma_{TE} = E_r/E_i$ is the reflection coefficient for *TE*-polarization. The total field in medium 1 is then given by the sum of the incident and reflected fields as $\tilde{E}_1(r,\omega) = \tilde{E}_i(r,\omega) + \tilde{E}_r(r,\omega)$ taken together with its associated magnetic field. The total field in medium 2 is given by the transmitted plane wave electric field vector $\hat{E}_t(r,\omega) = \hat{1}_y\tau_{TE}E_0 exp\left(i\tilde{k}_t(\omega)\cdot r\right)$ taken together with its associated magnetic field, with transmitted wave vector $\tilde{k}_t(\omega) = \vec{\beta}_t(\omega) + i\vec{\alpha}_t(\omega)$. Here

$$\vec{\beta}_t(\omega) = \beta_2(\omega)\left[\hat{1}_x \sin\Theta_t + \hat{1}_z\cos\Theta_t\right] \quad \& \quad \vec{\alpha}_t(\omega) = \hat{1}_z\alpha_2(\omega)\,, \tag{5}$$

with $\beta_2(\omega) = (\omega/c)n'_2(\omega)$, $\alpha_2(\omega) = (\omega/c)n''_2(\omega)$, where Θ_t is the angle of refraction, and where $\tau_{TE} = E_t/E_i$ is the transmission coefficient for *TE*-polarization.

Application of the tangential boundary conditions (2) then leads to the *law of reflection* $\Theta_r = \Theta_i$ as well as to the law of refraction

$$n'_1 \sin\Theta_i = n'_2 \sin\Theta_t \,, \tag{6}$$

provided that $n'_2 \geq n'_1 \sin\Theta_i$, which corresponds to subcritical incidence on the optically rarer medium. In addition, one obtains the *generalized Fresnel reflection and transmission coefficients for TE-polarization* as

$$\Gamma_{TE}(\omega) = \frac{\mu_2(n'_1\cos\Theta_i + in''_1) - \mu_1(n'_2\cos\Theta_t + in''_2)}{\mu_2(n'_1\cos\Theta_i + in''_1) + \mu_1(n'_2\cos\Theta_t + in''_2)} \,, \tag{6a}$$

$$\tau_{TE}(\omega) = \frac{2\mu_2(n'_1\cos\Theta_i + in''_1)}{\mu_2(n'_1\cos\Theta_i + in''_1) + \mu_1(n'_2\cos\Theta_t + in''_2)} \,, \tag{6b}$$

respectively, where $\tau_{TE} = 1 + \Gamma_{TE}$. Notice that both μ_1 and μ_2 may be complex-valued. These two expressions readily reduce to the well known idealized Fresnel coefficients when material loss is neglected as well as to the expressions derived by Besieris [3] when only one of the two media is lossy. At normal incidence ($\Theta_i = \Theta_t = 0$), these two coefficients simplify to the well-known expressions

$$\Gamma_{TE}(\omega)|_{\Theta_i=0} = \frac{\eta_2(\omega) - \eta_1(\omega)}{\eta_2(\omega) + \eta_1(\omega)} \,, \tag{7a}$$

$$\tau_{TE}(\omega)|_{\Theta_i=0} = \frac{2\eta_2(\omega)}{\eta_2(\omega) + \eta_1(\omega)} \,, \tag{7b}$$

where $\eta_j(\omega) = \left[\mu_j(\omega)/\epsilon_{cj}(\omega)\right]^{1/2}$ is the complex impedance of medium $j = 1,2$.

When incidence is on the optically rarer medium ($n'_1 > n'_2$), a *critical angle* of incidence exists that is given by

$$\Theta_c = \sin^{-1}(n'_2(\omega)/n'_1(\omega)) \,, \tag{8}$$

and is unaffected by the presence of any material loss. At both critical and supercritical angles of incidence $\Theta_i \geq \Theta_c$, $\Theta_t = \frac{\pi}{2}$ and the transmitted wave vector in medium 2 becomes

$$\tilde{k}_t(\omega) = \frac{\omega}{c}\left\{\hat{1}_x n'_2(\omega) + i\hat{1}_z\left[n''_2(\omega) + \sqrt{n'^2_1 sin^2(\Theta_i) - n'^2_2}\right]\right\} . \tag{9}$$

The *critical and supercritical Fresnel reflection and transmission coefficients for TE-polarization* are then found to be given by

$$\Gamma_{TE}(\omega) = \frac{\mu_2 n'_1\cos\Theta_i - i\left[\mu_1 n''_2 - \mu_2 n''_1 + \mu_1\sqrt{n'^2_1 sin^2(\Theta_i) - n'^2_2}\right]}{\mu_2 n'_1\cos\Theta_i + i\left[\mu_1 n''_2 + \mu_2 n''_1 + \mu_1\sqrt{n'^2_1 sin^2(\Theta_i) - n'^2_2}\right]} \,, \tag{10a}$$

$$\tau_{TE}(\omega) = \frac{2\mu_2(n'_1\cos\Theta_i + in''_1)}{\mu_2 n'_1\cos\Theta_i + i\left[\mu_1 n''_2 + \mu_2 n''_1 + \mu_1\sqrt{n'^2_1 sin^2(\Theta_i) - n'^2_2}\right]} \,, \tag{10b}$$

for $\Theta_i \geq \Theta_c$. For incidence at the critical angle $\Theta_i = \Theta_c = \sin^{-1}(n'_2/n'_1)$, both Eqs. (6) and (7) simplify to

$$\Gamma_{TE}(\omega)|_{\Theta_i=\Theta_c} = \frac{\mu_2\sqrt{n'^2_1 - n'^2_2} - i(\mu_1 n''_2 - \mu_2 n''_1)}{\mu_2\sqrt{n'^2_1 - n'^2_2} + i(\mu_1 n''_2 - \mu_2 n''_1)} \,, \tag{11a}$$

$$\tau_{TE}(\omega)|_{\Theta_i=\Theta_c} = \frac{2\mu_2\left(\sqrt{n'^2_1 - n'^2_2} + in''_1\right)}{\mu_2\sqrt{n'^2_1 - n'^2_2} + i(\mu_1 n''_2 - \mu_2 n''_1)} \,. \tag{11b}$$

These results show the dramatic influence that material loss has on the reflection and transmission properties of complex media. Although the value of the critical angle of incidence is found to be independent of the material loss [see Eq. (8)], the Fresnel coefficients are directly affected at all incidence angles, including critical and supercritical angles. In particular, the magnitude of the reflection coefficient at supercritical angles of incidence is reduced from its zero loss value of unity so that total reflection is no longer achieved unless $\mu_1 = \mu_2$ and medium 1 is lossless. These results may then be used, for example, to determine how much material loss can be tolerated over a given distance in a fiber optics communication link.

Similar results are obtained for *TM*-polarization. In that case, material loss is also found to have a significant effect on the appearance of a polarizing (Brewster) angle at which $\Gamma_{TM} = 0$. With the presence of material loss, the reflected TM-polarized wave field may not be completely suppressed. In addition, the polarizing angle is found to increase towards the critical angle as the loss in medium 2 increases.

4. References

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