32nd URSI GASS, Montreal, 19–26 August 2017

Spherical-Multipole Expansion of an Inhomogeneous Electromagnetic Plane Wave in Lossless Media

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Abstract

The spherical-multipole expansion of an inhomogeneous electromagnetic plane wave is derived by extending the spherical source coordinates ϑ to a complex number. The analytical results are successfully validated by comparing the numerical evaluation of the multipole expansion with the corresponding closed-form results. The paper also contains a first result for an inhomogeneous plane electromagnetic wave diffracted by a perfectly conducting semi-infinite circular cone.

1 Introduction

Inhomogeneous plane waves are well-known as they occur when a plane wave enters a lossy medium and the planes of constant phases and of constant amplitudes are no longer in parallel. More generally, they are also of interest for analyzing diffraction problems and for the mathematically complete representation of arbitrary fields by means of a superposition of plane waves [1], [2]. It has been shown that they can be obtained by simply introducing complex-valued coordinates. In [3] the technique has been used to investigate the diffraction of an inhomogeneous plane wave by a wedge. In that work the 2D eigenfunction solution of the wedge has been expressed in the form of the uniform theory of diffraction (UTD) so that it can be applied to compute the scattering from more complex structures. In the present work we follow this idea and successfully develop a 3D vector spherical-multipole expansion of an electromagnetic inhomogeneous plane wave. As will be proven, an inhomogeneous plane wave can be obtained by extending the spherical source coordinate ϑ to a complex number. Numerical results validate the analysis and show the convergence properties of the multipole expansion. Finally, a first result for the diffraction of an inhomogeneous electromagnetic plane wave by a perfectly conducting semi-infinite cone gives an outlook to a variety of interesting scattering problems.

2 Inhomogeneous plane wave in spherical coordinates

At a time factor $\exp(-j\omega t)$ the phasor of a plane wave is represented by

$$u(\vec{r}) = u_0 \exp(-j\vec{k}\cdot\vec{r}), \tag{1}$$

where u can denote a scalar plane wave, or the electric or magnetic field component of the plane electromagnetic wave at an arbitrary polarisation, as long as these components and the direction of propagation are perpendicular to each other. For an inhomogeneous plane wave with an exponential amplitude profile the vector wave number is complex-valued

$$\vec{k} = \vec{k}' - j\vec{k}'' \tag{2}$$

with k', k'' being real-valued. The direction of propagation of the phase front is given by $\vec{k}'$ while $\vec{k}''$ points into the direction of the exponential decay. As shown in [3], for a lossless medium $\vec{k}'$ and $\vec{k}''$ are perpendicular

$$\vec{k}'\cdot\vec{k}'' = 0 \tag{3}$$

and related to the real-valued wave number k by

$$(k')^2 - (k'')^2 = k^2 = \omega^2\varepsilon\mu\,. \tag{4}$$

Consequently it has been defined

$$k' = k\cosh\psi = k\cos(j\psi) \tag{5}$$
$$k'' = k\sinh\psi = -jk\sin(j\psi) \tag{6}$$

to properly satisfy (4).

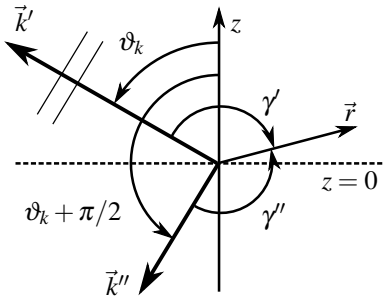

Figure 1. The real part $\vec{k}'$ and the imaginary part $\vec{k}''$ of the wave vector with the definition of the corresponding angles.

Now we use these properties to derive an inhomogeneous plane wave in spherical coordinates (r, ϑ, φ). A homogeneous plane wave is given by

$$u_h(\vec{r}) = u_0 \exp(-jkr\cos\gamma) \tag{7}$$
$$\cos\gamma = \cos\vartheta\cos\vartheta_k + \sin\vartheta\sin\vartheta_k\cos(\varphi-\varphi_k) \tag{8}$$

where γ is the angle between $\vec{k}$ and $\vec{r}$.

For an inhomogeneous plane wave the vectors $\vec{k}'$ and $\vec{k}''$ point into the directions (ϑ_k, φ_k) and $(\vartheta_k + \pi/2, \varphi_k)$, respectively, as shown in Figure 1. From (2) and (1) we derive with (7)

$$\begin{aligned} u_i(\vec{r}) &= u_0 \exp(-j(\vec{k}' - j\vec{k}'')\cdot\vec{r}) \\ &= u_0 \exp(-jk'r\cos\gamma')\exp(-k''r\cos\gamma'') \end{aligned} \tag{9}$$

where γ' is the angle between $\vec{r}$ and $\vec{k}'$ while γ'' is the angle between and $\vec{r}$ and $\vec{k}''$ (Fig. 1):

$$\cos\gamma' = \cos\vartheta\cos\vartheta_k + \sin\vartheta\sin\vartheta_k\cos(\varphi-\varphi_k) \tag{10}$$
$$\begin{aligned} \cos\gamma'' &= \cos\vartheta\cos(\vartheta_k+\pi/2) \\ &\quad + \sin\vartheta\sin(\vartheta_k+\pi/2)\cos(\varphi-\varphi_k) \\ &= -\cos\vartheta\sin\vartheta_k + \sin\vartheta\cos\vartheta_k\cos(\varphi-\varphi_k) \end{aligned} \tag{11}$$

Now inserting (10) and (11) into (9), utilizing (5) and (6), and performing some trigonometrics leads to:

$$\begin{aligned} u_i(\vec{r}) &= u_0 \exp(-jkr(\cos\vartheta\cos(\vartheta_k - j\psi) + \\ &\quad \sin\vartheta\sin(\vartheta_k - j\psi)\cos(\varphi-\varphi_k))). \end{aligned} \tag{12}$$

A comparison with (7) and (8) reveals that the inhomogeneous plane wave (12) is exactly the representation of a homogeneous plane wave in spherical coordinates but with a complex-valued γ, more precisely, a complex-valued $\vartheta_k - j\psi$.

We conclude that simply choosing a complex-valued angle $\vartheta_c = \vartheta_k - j\psi$ instead of ϑ_k, a homogeneous plane wave is transformed to an inhomogeneous plane wave with an exponential profile. According to (6) ϑ_c is given by

$$\vartheta_c = \vartheta_k - j\psi = \vartheta_k - j\,\mathrm{arcsinh}\left(\frac{k''}{k}\right). \tag{13}$$

Now, ϑ_k and φ_k depict the direction of propagation of the plane wave's phase front, while the damping factor k'' describes the exponential decay according to $\exp(-\vec{k}''\cdot\vec{r})$ in the direction perpendicular to the direction of propagation

$$\hat{k}'' = \hat{x}\cos\vartheta_k\cos\varphi_k + \hat{y}\cos\vartheta_k\sin\varphi_k - \hat{z}\sin\vartheta_k \tag{14}$$

with the hat symbol denoting the unit vector.

3 Vector spherical-multipole expansion of an inhomogeneous plane wave

The spherical-multipole expansion of an electromagnetic plane wave reads [4]

$$\vec{E} = \sum_{n=0}^{\infty}\sum_{m=-n}^{+n} a_{n,m}\vec{N}^I_{n,m} + \frac{Z}{j}\sum_{n=0}^{\infty}\sum_{m=-n}^{+n} b_{n,m}\vec{M}^I_{n,m} \tag{15}$$
$$\vec{H} = \frac{j}{Z}\sum_{n=0}^{\infty}\sum_{m=-n}^{+n} a_{n,m}\vec{M}^I_{n,m} + \sum_{n=0}^{\infty}\sum_{m=-n}^{+n} b_{n,m}\vec{N}^I_{n,m}. \tag{16}$$

Here, $Z = \sqrt{\mu/\varepsilon}$ is the intrinsic impedance while the expansion functions which are referred to as the vector spherical-multipole functions $\vec{M}^I$ and $\vec{N}^I$ are given by

$$\vec{M}^I_{n,m}(r,\vartheta,\varphi) = j_n(\kappa r)\,\vec{m}_{n,m}(\vartheta,\varphi) \tag{17}$$
$$\begin{aligned} \vec{N}^I_{n,m}(r,\vartheta,\varphi) = &-\frac{j_n(\kappa r)}{\kappa r}\,n(n+1)Y_{n,m}(\vartheta,\varphi)\hat{r} \\ &-\frac{1}{\kappa r}\frac{d}{dr}[r j_n(\kappa r)]\vec{n}_{n,m}(\vartheta,\varphi). \end{aligned} \tag{18}$$

In (17) and (18) j_n denotes a spherical Bessel function of the first kind while the normalized surface spherical harmonics $Y_{n,m}$ are defined by

$$Y_{n,m} = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\vartheta)\exp(jm\varphi) \tag{19}$$

with $P_n^m(\cos\vartheta)$ being associated Legendre functions of the first kind. The transverse vector functions are defined by

$$\begin{aligned} \vec{m}_{n,m}(\vartheta,\varphi) = &-\frac{1}{\sin\vartheta}\frac{\partial Y_{n,m}(\vartheta,\varphi)}{\partial\varphi}\hat{\vartheta} \\ &+\frac{\partial Y_{n,m}(\vartheta,\varphi)}{\partial\vartheta}\hat{\varphi} \end{aligned} \tag{20}$$
$$\begin{aligned} \vec{n}_{n,m}(\vartheta,\varphi) = &\frac{\partial Y_{n,m}(\vartheta,\varphi)}{\partial\vartheta}\hat{\vartheta} \\ &+\frac{1}{\sin\vartheta}\frac{\partial Y_{n,m}(\vartheta,\varphi)}{\partial\varphi}\hat{\varphi}. \end{aligned} \tag{21}$$

The expansion coefficients are referred to as the multipole amplitudes $a_{n,m}$ and $b_{n,m}$. They contain all of the problem-specific field information - in our case of the inhomogeneous plane wave. The amplitudes of a homogeneous plane wave incident from ϑ_k, φ_k and electrically polarized in the direction $\hat{E}$ are given by [4]

$$a_{n,m} = E_0 4\pi (j)^{n+1}\frac{(-1)^m}{n(n+1)}\vec{n}_{n,-m}(\vartheta_k,\varphi_k)\cdot\hat{E} \tag{22}$$
$$b_{n,m} = \frac{E_0}{Z_0}4\pi (j)^{n+1}\frac{(-1)^m}{n(n+1)}\vec{m}_{n,-m}(\vartheta_k,\varphi_k)\cdot\hat{E}. \tag{23}$$

For an inhomogeneous plane wave, we simply have to replace ϑ_k by ϑ_c according to (13).

Apparently, when inserting a complex-valued ϑ_c, we need to calculate associated Legendre functions of the first kind $P_n^m(\cos\vartheta_c)$ for a complex-valued argument which turns out

to be a rather delicate task. We have successfully used a representation with hypergeometric functions which has been proven in [5] to be well suited for the calculation of the associated Legendre functions for complex-valued arguments.

4 Numerical evaluation

To numerically validate the results we have first evaluated the spherical-multipole expansion for an angle of incidence $\vartheta_k = \pi/2$ and $\varphi_k = 0$, i.e., the plane wave is propagating in the negative x-direction, damped in the negative z-direction, and electrically polarized in the y-direction. We compare the multipole-expansion results for different upper limits n_{max} of the order n and for different imaginary parts ψ to those ones obtained by the corresponding closed-form representations.

For a first overview on the convergence properties of the multipole expansion we see in Figure 2 the relative error for an upper limit $n_{max} = 20$ and $\psi = 0.07\lambda$ in the plane $y = 0$. As expected for a spherical expansion, due to the behaviour of the spherical Bessel functions of the first kind it converges well near the origin while the accuracy decreases with an increasing r. In Figure 3 the relative error is com-

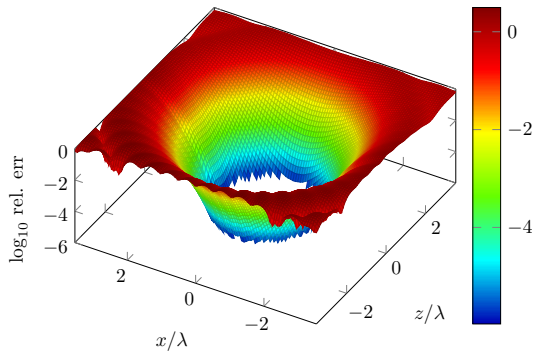

Figure 2. Relative error of the vector spherical-multipole expansion (maximum order $n_{max} = 20$) of an electromagnetic inhomogeneous plane wave($\psi = 0.07\lambda$) in the xz-plane.

puted for different n_{max} along the z-axis. It can be seen, that the convergence properties are slightly better in positive z-direction than in negative z-direction. Furthermore we note that the error stays roughly in a radius of 3λ below the 10^{-3} threshold for $n_{max} = 40$. The convergence is slightly poorer for higher values of ψ as we can see in Fig. 4 where n_{max} is 20 and $\psi = 0$ (i.e. homogeneous plane wave), $\psi = 0.07\lambda$ and $\psi = 0.14\lambda$. Fig. 5 shows a snap-shot of the electric field (i.e., the real part of the phasor) in the xz-plane. We clear identify the plane wave with the desired exponential damping perpendicularly to the phase fronts. To initially demonstrate the potential of this new approach Figure 6 shows a snap shot of the electric field of an inhomogeneous plane wave travelling in $-x$ direction and diffracted by a perfectly conducting circular cone. The eigenvalues for the corresponding spherical-multipole

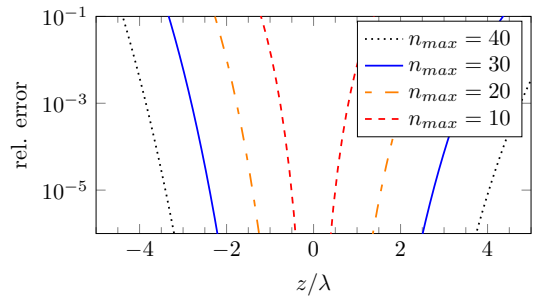

Figure 3. Relative error of the vector spherical-multipole expansion for different maximum orders n_{max} of an electromagnetic inhomogeneous plane wave ($\psi = 0.07\lambda$) in the xz-plane along the z-axis.

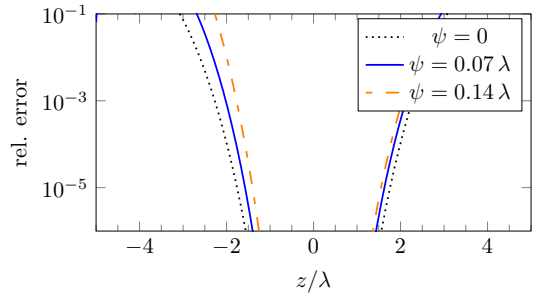

Figure 4. Relative error of the vector spherical-multipole expansion for different maximum orders n_{max} of an electromagnetic inhomogeneous plane wave ($\psi = 0.14\lambda$) in the xz-plane along the z-axis.

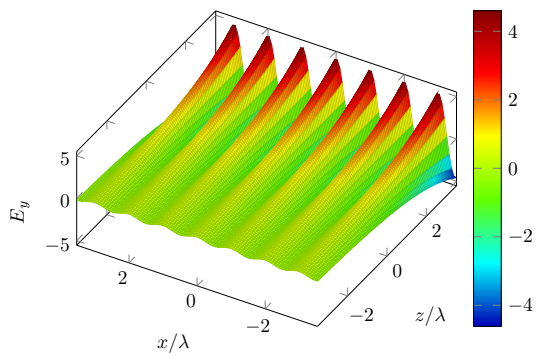

Figure 5. Snap-shot of the electric field of the vector spherical-multipole expansion (maximum order $n_{max} = 40$) of an electromagnetic inhomogeneous plane wave ($\psi = 0.07\lambda$) in the xz-plane.

expansion are found by a numerical approach according to [6]. For $x > 0$ we identify the interference of the incident field with the field reflected by the cone while in the shadow region of the cone (for $x < 0$) we observe the field diffracted by the sides and by the tip of the cone.

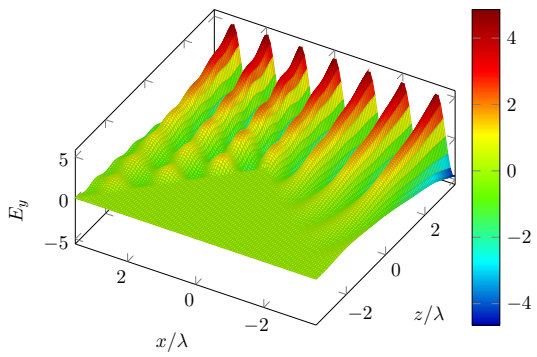

Figure 6. Snap-shot of the electric field of the vector spherical-multipole expansion (maximum order $\nu_{max} \approx 40$) of an electromagnetic inhomogeneous plane wave ($\psi = 0.07\lambda$) in the xz-plane diffracted a circular cone located symmetrically around the $-z$-axis.

5 Conclusion

We have demonstrated that a simple extension of the spherical coordinate ϑ to a complex-valued number leads to an inhomogeneous plane wave which can be successfully used in the spherical-multipole expansion of an inhomogeneous electromagnetic plane wave. The corresponding convergence analysis shows some results which must be further investigated. Future work also includes the further investigation of canonical scattering problems and the derivation of corresponding diffraction coefficients.

References

[1] P.C. Clemmow, *The plane wave spectrum representation of electromagnetic fields*, Tarrytown: Pergamon Press, 1980.

[2] M. Born and E. Wolf, *Principles of optics (5th ed.)*, Oxford: Pergamon Press, 1975.

[3] R. G. Kouyoumjian, G. Manara, P. Nepa, B. J. E. Taute, "The diffraction of an inhomogeneous plane wave by a wedge," *Radio Science*, **31**, 6, Nov./Dec. 1996, pp. 1387-1397, doi: 10.1109/8.736640

[4] S. Blume and L. Klinkenbusch, " Spherical-multipole analysis in electromagnetics", in Werner, D., and R. Mittra (ed.), *Frontiers in Electromagnetics*, 553, IEEE Press 2000.

[5] A. Reinhardt, H. Bruens, L. Klinkenbusch, M. Katsav, E. Heyman, "Spherical-multipole analysis of an arbitrarily directed complex-source beam diffracted by an acoustically soft or hard circular cone", *Advances in Radio Science*, Vol. 13, pp. 57–61, Nov. 2015, doi: 10.5194/ars-13-57-2015.

[6] S. Blume, V. Krebs: "Der Rückstreuquerschnitt des semiinfiniten Kreiskegels (The backscattering cross section of the semi-infinite circular cone)", Arch. Elektrotech., 77, 239-244, 1994