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On Some Mathematical Aspects of High-frequency Diffraction by Strongly Elongated Spheroids

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Abstract

The high-frequency asymptotic procedure developed for the diffraction by an elongated body is proved to reduce the initial problem to a mathematically correct boundary-value problem for the parabolic equation. The analogue of "time" η varies on an interval $[-1,1]$. The problem is considered in the class of functions with not too singular behavior near $\eta=\pm 1$ and the solution is understood in the frame of limiting absorption principle. Uniqueness theorem is proved and the explicit representation for the solution is checked to satisfy the required properties.

1 Introduction

The classical parabolic equation method [1] when applied to problems of high-frequency diffraction by strongly elongated bodies meets some difficulties because of the competition between the large asymptotic parameter and the rate of elongation. This results in the underestimation of the induced currents magnitude in the light-shadow transition region. The special asymptotic approach based on the use of the parabolic equation method in the scaled spheroidal coordinates was suggested in [2] and then further developed in [3, 4, 5] and other papers. The derivations in these publications were carried out formally, which is usually accepted in diffraction theory, and therefore require mathematical justification. The goal of the present paper is to justify some issues of the asymptotic procedure. We do that for scalar problem.

2 Reducing the Problem to Parabolic Equation

We consider diffraction of a plane harmonic wave of frequency ω incident at a small angle θ to the axis of prolate spheroid $(x^2+y^2)a^{-2}+z^2b^{-2}=1$ with the semiaxes b and a, $b>a$. The time factor is assumed to be $e^{-i\omega t}$ and is omitted elsewhere. The scattered field U satisfies Helmholtz equation

$$\triangle U + k^2 U = 0 \qquad (1)$$

at the exterior of the spheroid. The total field $U+U^i$, where U^i is the incident wave

$$U^i = \exp\left(ikx\sin\theta + ikz\cos\theta\right), \qquad (2)$$

satisfies the Dirichlet boundary conditions on the surface and U is subject to the limiting absorption condition at infinity.

We introduce the focal distance $p=\sqrt{b^2-a^2}$ and consider the case $kp \gg 1$. The elongation of the spheroid is characterized by the parameter $\chi=(ka)^2/(kp)$, which we assume on the order of one. Besides we assume that the angle θ is small, so that $\alpha \equiv \sqrt{kp}\,\theta = O(1)$. The last two assumptions allow to consider the problem of diffraction as the problem with a single asymptotic parameter kp.

We search for the scattered field in the form of Fourier series

$$U(x,y,z) = \sum_{n=0}^{+\infty} U_n(r,z)\cos(n\phi) \qquad (3)$$

in cylindrical coordinates (r,z,ϕ). This reduces the problem to independent harmonics.

Further we introduce spheroidal coordinates (η,ξ), such that

$$r = p\sqrt{1-\eta^2}\sqrt{\xi^2-1}, \quad z = p\eta\xi. \qquad (4)$$

In view of the assumption that $\chi=O(1)$ the coordinate ξ in a vicinity of the surface takes values close to 1. Therefore, we scale it by introducing τ:

$$\xi = 1 + \frac{\chi}{kp}\tau. \qquad (5)$$

The coefficient in (5) is chosen such that $\tau=1$ on the surface of the spheroid.

The asymptotic representation of each U_n is searched according to the parabolic equation method in the form

$$U_n = e^{ikp\eta}\sum_{j=0}^{\infty}\frac{u_{n,j}(\eta,\tau)}{(kp)^j}, \qquad (6)$$

where we assume the dependence of $u_{n,j}$ on η and τ to be such that differentiation with respect to these arguments does not change the asymptotic order of terms with respect to kp.

By substituting the representation (6) into the Helmholtz equation, which should be first rewritten in coordinates η,τ, we get the recurrent system of equations

$$L_n u_{n,j} + N_n u_{n,j-1} = 0, \qquad (7)$$

where

$$L_n = \tau\frac{\partial^2}{\partial\tau^2} + \frac{\partial}{\partial\tau} + \frac{i\chi\left(1-\eta^2\right)}{2}\frac{\partial}{\partial\eta} - \frac{i\chi\eta}{2} - \frac{n^2}{4\tau} + \frac{\chi^2\tau}{4}, \quad (8)$$

$$N_n = \chi^2\tau^2\frac{\partial^2}{\partial\tau^2} + \frac{3}{2}\chi^2\tau\frac{\partial}{\partial\tau} + \frac{\chi}{4}\left(1-\eta^2\right)\frac{\partial^2}{\partial\eta^2} - \frac{\chi}{2}\eta\frac{\partial}{\partial\eta} - \frac{\chi}{4}\frac{n^2}{1-\eta^2}.$$

We restrict our analysis to the leading order terms $u_{n,0}$ only, for which the equation (7) is homogeneous. Below we discard the index zero, that is u_n stands further for $u_{n,0}$.

3 Main Results

The boundary value problem at the leading order consists of the parabolic equation

$$\tau\frac{\partial^2 u_n}{\partial\tau^2} + \frac{\partial u_n}{\partial\tau} + \frac{i\chi\left(1-\eta^2\right)}{2}\frac{\partial u_n}{\partial\eta} + \left[\frac{\chi^2\tau}{4} - \frac{n^2}{4\tau} - \frac{i\chi\eta}{2}\right]u_n = 0, \quad (9)$$

and the boundary condition

$$u_n(\eta,1) = -u_n^i(\eta,\tau), \quad (10)$$

where the right-hand side is the leading order term of the incident wave asymptotics:

$$u_n^i = 2\,i^n\exp\left(\frac{i}{2}\left(\chi\tau-\alpha^2\right)\eta\right)J_n\left(\sqrt{\chi\tau}\alpha\sqrt{1-\eta^2}\right). \quad (11)$$

We consider the boundary-value problem (9), (10) in the class of functions for which we set the restriction on the possible singularities at $\eta\to\pm1$:

$$u_n\in L_2(\Omega_0), \qquad (1-\eta^2)\frac{\partial u_n}{\partial\eta}\in L_2(\Omega_0), \quad (12)$$

where Ω_0 is arbitrary small neighborhood of the points $\eta=\pm1$, and we also require a sufficiently quick decay of the solution with respect to $\tau\to+\infty$ in the case of $\Im\chi>0$. The solution for real χ is understood as the limit as $\Im\chi\to0$.

The last requirement can be formalized as

$$u_n\in\sqrt{\tau}u_n,\in L_2, \qquad \sqrt{\tau}\frac{\partial u_n}{\partial\tau}\in L_2. \quad (13)$$

Analysis of the problem allows the following results to be obtained.

Theorem 1 *For a medium with damping, the BVP (9), (10) in the class (12) and (13) has a unique solution.*

Theorem 2 *The solution to the BVP (9), (10), (11) in the class (12) and (13) can be represented in the form of the integral involving the Whittaker functions M and W :*

$$u_n = \frac{-2\,i^n}{\pi(n!)^2\sqrt{1-\eta^2}}\int\limits_{-\infty}^{+\infty}\left(\frac{1-\eta}{1+\eta}\right)^{it}\left|\Gamma\left(\tfrac{n+1}{2}+it\right)\right|^2 \times\frac{M_{it,n/2}(i\alpha^2)}{\alpha}\frac{W_{it,n/2}(-i\chi\tau)}{\sqrt{\chi\tau}}\frac{M_{it,n/2}(-i\chi)}{W_{it,n/2}(-i\chi)}dt. \quad (14)$$

Theorem 3 *(a) For every $R\geq1$, the attenuation function u given by Fourier series $u=\sum_{n=0}^{+\infty}u_n\cos(n\phi)$ belongs to $C^\infty(\Theta_R)$, where $\Theta_R=\{(\eta,\tau,\phi)|\eta\in(-1,1),\ \tau\in[1,R),\ \phi\in[0,2\pi)\}$.*

(b) For $\Im\chi>0$, the attenuation function u decays exponentially as $\tau\to\infty$.

The proof of Theorem 1 is based on the scheme of [6]. That is, we make the boundary condition to be homogeneous and get the nonzero right-hand side F in (9). Then we introduce the set H as the closure of all smooth (complex valued) functions u satisfying the conditions (12) and the boundary condition $u|_{\tau=1}=0$ in the norm

$$\|u\|^2 = \iint\left(\tau|u_\tau|^2 + \frac{n^2}{4\tau}|u|^2 + \delta\cdot\tau|u|^2\right)d\eta\,d\tau \quad (15)$$

with some fixed $\delta>0$. We note that the set H becomes the Hilbert space if we introduce the inner product according to (15)

$$[u,v] = \iint\left(\tau u_\tau\overline{v}_\tau + \frac{n^2}{4\tau}u\overline{v} + \delta\cdot\tau u\overline{v}\right)d\eta\,d\tau.$$

We define the weak solution V of the boundary-value problem

$$L_nV = F$$

as an element $V\in H$ that satisfies, for every test function $w\in H$ the identity

$$-\iint\tau V_\tau\overline{w}_\tau d\eta\,d\tau + \frac{i\chi}{2}\iint(1-\eta^2)V_\eta\overline{w}d\eta\,d\tau + \frac{\chi^2}{4}\iint\tau V\overline{w}d\eta\,d\tau - \frac{n^2}{4}\iint\frac{1}{\tau}V\overline{w}d\eta\,d\tau - \frac{i\chi}{2}\iint\eta V\overline{w}d\eta\,d\tau = \iint F\overline{w}d\eta\,d\tau, \quad (16)$$

where subscripts denote derivatives.

Further, we suppose that two different solutions V_1 and V_2 exist and consider their difference $V=V_1-V_2$, for which $F=0$. We substitute $w=V$ in (16) and separate the imaginary part. Then by integrating by parts in (16) we come to a contradiction.

The result of Theorem 2 is based on the use of the integral transform [7]

$$F(\eta) = \frac{1}{\sqrt{1-\eta^2}} \int_{-\infty}^{+\infty} \left(\frac{1-\eta}{1+\eta}\right)^{it} \mathscr{P}[F](t)dt, \qquad (17)$$

$$\mathscr{P}[F](s) = \frac{1}{\pi} \int_{-1}^{1} \left(\frac{1+\eta}{1-\eta}\right)^{is} \frac{F(\eta)}{\sqrt{1-\eta^2}} d\eta, \qquad (18)$$

which reduces the equation (9) to an ordinary differential equation. We represent its solution in terms of Whittaker functions and use their asymptotic properties to check that the integrals converge exponentially as $t \to \pm\infty$. The analysis of the right-hand side of (14) when $\eta \to \pm 1$ requires the study of the zeros t of the dispersion equations

$$W_{it,n/w}(-i\chi) = 0, \qquad (19)$$

for which we proved that

$$\Im t < -\frac{1}{2}. \qquad (20)$$

For the proof of Theorem 3, the crucial estimate

$$\left| \frac{W_{it,n/2}(-i\chi\tau)}{W_{it,n/2}(-i\chi)} \right|^2 \leq \tau e^{\Im\chi(1-\tau)} \qquad (21)$$

assuming $\Im t = 0$, $\tau \geq 1$, $\Im\chi \geq 0$, was obtained by H. W. Volkmer in [8]. This estimate enables us to prove uniform convergence of the Fourier series for u by Weierstrass M–test.

4 Generalizations

The above results allow some generalizations. In particular Theorem 1 remains valid for the case of Neumann boundary condition. In formula (14) one need to change the last factor to

$$\frac{M_{it,n/2}(-i\chi) + 2i\chi \dot{M}_{it,n/2}(-i\chi)}{W_{it,n/2}(-i\chi) + 2i\chi \dot{W}_{it,n/2}(-i\chi)},$$

where dot denotes derivative of the function. With that change Theorem 2 becomes true for the case of the Neumann boundary condition.

The only difficulty that remains to overcome in the case of the diffraction problem with the Neumann boundary condition is to prove the result of Theorem 3. We have not done this yet, because instead of (21) we need to estimate the ratio

$$\left| \frac{W_{it,n/2}(-i\chi\tau) + 2i\chi \dot{W}_{it,n/2}(-i\chi\tau)}{W_{it,n/2}(-i\chi) + 2i\chi \dot{W}_{it,n/2}(-i\chi)} \right|. \qquad (22)$$

The problem of diffraction of an electromagnetic field by a perfectly conducting spheroid is also reduced to parabolic equations. However, in the electromagnetic case there is a system of two such equations for each harmonics. This system can be split if the new unknowns P_n and Q_n are introduced as

$$E_n^\phi = Q_n + P_n, \quad iH_n^\phi = Q_n - P_n,$$

where E^ϕ and H^ϕ are the transverse components of the electric and magnetic fields. At the leading order with respect to kp functions Q_n and P_n satisfy the parabolic equations

$$L_{n-1}Q_n = 0, \qquad L_{n+1}P_n = 0. \qquad (23)$$

Therefore uniqueness Theorem remains true.

The expression for the scattered field becomes more complicated (we do not present here the formulas, which can be found in [5]). It is possible to justify the derivations and prove convergence of the integrals. However, the analysis of the representations for P_n and Q_n as $\eta \to \pm 1$ requires the zeros of the dispersion equation

$$W_{it,n+1/2}(-i\chi)\dot{W}_{it,n-1/2}(-i\chi) + \dot{W}_{it,n+1/2}(-i\chi)W_{it,n-1/2}(-i\chi) = 0 \qquad (24)$$

to be studied.

Further, one needs to establish uniform convergence of the Fourier series for the attenuation functions, which is required for the proof of Theorem 3. This will be the subject of further research.

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