32nd URSI GASS, Montreal, 19–26 August 2017

Analysis of a Strip Antenna Located on the Interface Between an Isotropic Medium and a Hyperbolic Metamaterial

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Abstract

Electrodynamic characteristics of a straight strip antenna located on the plane interface between an isotropic medium and a hyperbolic metamaterial are studied using the integral equation method. A closed-form solution for the current distribution of an infinitely long strip that is perpendicular to the anisotropy axis of the metamaterial is obtained and the input impedance of such an antenna is found. Based on this solution, a generalization to the case of a finite-length strip antenna is discussed.

1 Introduction

Over the past decade there has been a substantial degree of interest in the excitation and propagation of electromagnetic waves in systems containing hyperbolic metamaterials (see, e.g., [1] and references therein). Such materials are a type of resonant media in which at least one of the normal waves possesses an unbounded branch of the refractive index surface having a hyperbolic shape. This feature of the wave dispersion resembles that occurring in a resonant magnetoplasma [2], thereby leading to similar difficulties when developing antenna theory [3]. Despite numerous works on the subject, a satisfactory solution to the problem of finding the electrodynamic characteristics of wire antennas operated in the presence of hyperbolic metamaterials is yet to be found. In this work, we study the current distribution and input impedance of a straight strip antenna located on the interface between an isotropic medium and a hyperbolic metamaterial whose permittivity and permeability tensors are typical of uniaxial anisotropy.

Although our approach is applicable to a general uniaxial medium on the one side of the interface, we focus on the most interesting case of a hyperbolic metamaterial in which the refractive indices of both normal waves tend to infinity at certain angles between the wave vector and the anisotropy axis. Note that a solution of the antenna problem in this case is of both practical and academic interest.

2 Formulation of the Problem

Consider a straight antenna of infinite length, which is aligned with the x axis and lies in the xz plane (see Fig. 1). The antenna having the form of an infinitesimally thin, perfectly conducting narrow strip of half-width d is located on the interface between an isotropic medium and a uniaxial metamaterial. The half-space $y < 0$ is filled with a metamaterial that is assumed lossless and describable by the permittivity and permeability tensors

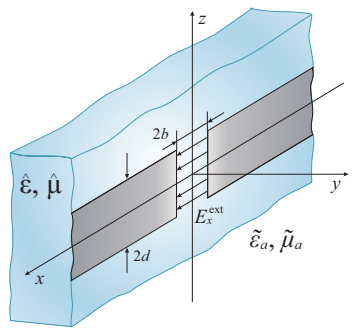

Figure 1. Geometry of the problem.

$$\hat{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_\perp & 0 & 0 \\ 0 & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_\parallel \end{pmatrix}, \ \hat{\mu} = \mu_0 \begin{pmatrix} \mu_\perp & 0 & 0 \\ 0 & \mu_\perp & 0 \\ 0 & 0 & \mu_\parallel \end{pmatrix},$$

where ε_0 and μ_0 are the permittivity and permeability of free space, respectively. The anisotropy axis of such a metamaterial is aligned with the z axis. The elements of the tensors are functions of the angular frequency ω and depend on a particular structure of the medium [4]. We restrict ourselves to consideration only of the case where the following conditions are satisfied simultaneously for the diagonal elements of the tensors:

$$\operatorname{sgn}\varepsilon_\perp \neq \operatorname{sgn}\varepsilon_\parallel, \tag{1}$$

$$\operatorname{sgn}\mu_\perp \neq \operatorname{sgn}\mu_\parallel. \tag{2}$$

The medium occupying another half-space ($y > 0$) is isotropic and has the permittivity $\tilde{\varepsilon}_a = \varepsilon_0\varepsilon_a$ and the permeability $\tilde{\mu}_a = \mu_0\mu_a$.

The current on the strip surface is excited by a time-harmonic ($\sim \exp(i\omega t)$) voltage that is applied over a narrow interval $|x| \leq b$ and creates an electric field with the only nonzero component $E_x^{\rm ext}$ in this interval on the surface of the strip (i.e., at $y = 0$ and $|z| < d$):

$$E_x^{\rm ext} = \frac{V_0}{2b}\left[U(x+b) - U(x-b)\right]\left[U(z+d) - U(z-d)\right]. \tag{3}$$

Here, $V_0 = \text{const}$ is a constant amplitude of the given voltage, U is a Heaviside function, and b is the half-width of the antenna excitation gap.

3 Derivation of the Integral Equations for the Antenna Current

Using the Fourier transform technique, we represent excitation field (3) for $|z| < d$ in the form

$$E_x^{\text{ext}}(x,0,z) = \frac{k_0}{2\pi}\int_{-\infty}^{\infty} \mathscr{E}_x^{\text{ext}}(n_x)\exp(-ik_0 n_x x)dn_x,$$

where $k_0 = \omega(\varepsilon_0\mu_0)^{1/2}$ is the wave number in free space and $\mathscr{E}_x^{\text{ext}}(n_x) = V_0\sin(k_0 n_x b)/(k_0 n_x b)$. The density $\mathbf{J}$ of the electric current excited on the antenna by the field E_x^{ext} can be sought as

$$\mathbf{J} = \mathbf{x}_0 I(x,z)\delta(y),$$

where $|z| < d$, δ is a Dirac function, and $I(x,z)$ is the surface current density which admits the following representation:

$$I(x,z) = \frac{k_0}{2\pi}\int_{-\infty}^{\infty} \mathscr{I}(n_x,z)\exp(-ik_0 n_x x)dn_x. \qquad (4)$$

Allowing for (4) and satisfying the radiation conditions at infinity and the boundary conditions for the tangential field components at the interface $y = 0$, we express the antenna-excited field components $E_x(\mathbf{r})$ and $E_z(\mathbf{r})$ in the half-spaces $y < 0$ and $y > 0$ in terms of Fourier integrals over the normalized (to k_0) wave numbers n_x and n_z. After some algebra, we can write these field components at $y = 0$ as

$$\begin{bmatrix} E_x(x,0,z) \\ E_z(x,0,z) \end{bmatrix} = \frac{k_0}{2\pi}\int_{-\infty}^{\infty} dn_x \int_{-d}^{d} \begin{bmatrix} K_x(n_x, z-z') \\ K_z(n_x, z-z') \end{bmatrix} \times \mathscr{I}(n_x,z')\exp(-ik_0 n_x x)dz'. \qquad (5)$$

Here,

$$\begin{bmatrix} K_x(n_x,\zeta) \\ K_z(n_x,\zeta) \end{bmatrix} = \frac{Z_0 k_0}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} -(n_x n_z B_1 + n_{y,2}\mu_\perp B_2)\,\kappa^{-2} \\ B_1 \operatorname{sgn}\zeta \end{bmatrix} \times \Delta^{-1}(n_x,n_z)\exp(-ik_0 n_z|\zeta|)dn_z, \qquad (6)$$

where Z_0 is the impedance of free space. Other quantities in (6) are defined as

$$B_1 = \frac{n_x n_z}{\kappa^2 n_\perp^2}(n_{y,2}\mu_\perp + n_y\mu_a), \quad B_2 = \frac{\delta_1 n_y \mu_a + \beta(n_x n_z)^2}{n_\perp^2},$$
$$\Delta(n_x,n_z) = \delta_1\delta_2 + \beta^2(n_x n_z)^2, \quad \beta = n_\perp^{-2} - \kappa^{-2},$$
$$\delta_1 = \varepsilon_a\frac{n_y}{n_\perp^2} + \varepsilon_\perp\frac{n_{y,1}}{\kappa^2}, \quad \delta_2 = \mu_a\frac{n_y}{n_\perp^2} + \mu_\perp\frac{n_{y,2}}{\kappa^2},$$
$$n_\perp^2 = \varepsilon_a\mu_a - n_z^2, \quad n_y = \left(n_\perp^2 - n_x^2\right)^{1/2},$$
$$\kappa^2 = \varepsilon_\perp\mu_\perp - n_z^2, \quad n_{\perp 1}^2 = \frac{\varepsilon_\parallel}{\varepsilon_\perp}\kappa^2, \quad n_{\perp 2}^2 = \frac{\mu_\parallel}{\mu_\perp}\kappa^2,$$
$$n_{y,k} = \left(n_{\perp k}^2 - n_x^2\right)^{1/2}, \quad k = 1,2.$$

In the above formulas, n_y and $n_{y,k}$ are defined so as to ensure the conditions $\operatorname{Im} n_y < 0$ and $\operatorname{Im} n_{y,k} < 0$.

Using the boundary conditions $E_x = -E_x^{\text{ext}}$ and $E_z = 0$ for the tangential electric-field components on the strip surface ($y = 0$ and $|z| < d$), we obtain the integral equations

$$\int_{-d}^{d} K_x(n_x, z-z')\mathscr{I}(n_x,z')dz' = -\mathscr{E}_x^{\text{ext}}(n_x), \qquad (7)$$

$$\int_{-d}^{d} K_z(n_x, z-z')\mathscr{I}(n_x,z')dz' = 0 \qquad (8)$$

for the Fourier transform $\mathscr{I}(n_x,z)$ of the surface current density, where $|z| < d$.

4 Solution of the Integral Equations

The behavior of solutions of the obtained integral equations is determined by the properties of their kernels. We can represent the kernels $K_x(n_x,\zeta)$ and $K_z(n_x,\zeta)$ of integral equations (7) and (8) as the sums of singular and regular parts:

$$K_{x,z}(n_x,\zeta) = K_{x,z}^{(s)}(n_x,\zeta) + K_{x,z}^{(r)}(n_x,\zeta).$$

The singular parts $K_x^{(s)}(n_x,\zeta)$ and $K_z^{(s)}(n_x,\zeta)$ tend to infinity for $\zeta \to 0$, whereas the regular parts $K_x^{(r)}(n_x,\zeta)$ and $K_z^{(r)}(n_x,\zeta)$ remain finite in this limit and can be taken at $\zeta = 0$ if the antenna is so narrow that the following conditions take place:

$$d \ll 2b,$$
$$(k_0 d)^2 \max\{|\varepsilon_a|, |\varepsilon_\perp|, |\varepsilon_\parallel|, |\mu_a|, |\mu_\perp|, |\mu_\parallel|\} \ll 1. \qquad (9)$$

With allowance for conditions (9), the singular parts of the kernels $K_x(n_x,\zeta)$ and $K_z(n_x,\zeta)$ in the limit $\zeta \to 0$ are found to have the logarithmic and Cauchy singularities, respectively:

$$K_x^{(s)}(n_x,\zeta) = -\frac{iZ_0 k_0}{\pi}\left\{\left(\ln\frac{k_0|\zeta|}{2} + \ln|n_x| + \gamma\right) \times \left(\frac{i|\mu_\perp\mu_\parallel|^{1/2}\mu_a}{\mu_a - i|\mu_\perp\mu_\parallel|^{1/2}} + \frac{n_x^2}{\varepsilon_a - i|\varepsilon_\perp\varepsilon_\parallel|^{1/2}}\right) - \frac{\mu_a|\mu_\perp\mu_\parallel|}{\mu_a^2 + |\mu_\perp\mu_\parallel|}\ln\sqrt{\frac{|\mu_\perp|}{|\mu_\parallel|}} + \frac{in_x^2|\varepsilon_\perp\varepsilon_\parallel|^{1/2}}{\varepsilon_a^2 + |\varepsilon_\perp\varepsilon_\parallel|}\ln\sqrt{\frac{|\varepsilon_\perp|}{|\varepsilon_\parallel|}}\right\},$$

$$K_z^{(s)}(n_x,\zeta) = \frac{Z_0}{\pi}\frac{n_x}{\varepsilon_a - i|\varepsilon_\perp\varepsilon_\parallel|^{1/2}}\frac{1}{\zeta},$$

where $\gamma = 0.5772\ldots$ is Euler's constant. As a result, integral equations (7) and (8) take the form

$$\int_{-d}^{d}\mathscr{I}(n_x,z')\ln\frac{k_0|z-z'|}{2}dz' = -\frac{2i\pi}{Z_0 k_0}\chi\mathscr{E}_x^{\text{ext}}(n_x) - S(n_x)\int_{-d}^{d}\mathscr{I}(n_x,z')dz', \qquad (10)$$

$$\int_{-d}^{d} n_x \frac{\mathscr{I}(n_x, z')}{z - z'} dz' = 0, \tag{11}$$

where

$$\chi = \frac{\varepsilon_{\text{eff}}}{n_x^2 - \varepsilon_{\text{eff}}\mu_{\text{eff}}}.$$

Here,

$$\varepsilon_{\text{eff}} = \frac{\varepsilon_a - i|\varepsilon_\perp \varepsilon_\parallel|^{1/2}}{2}, \quad \mu_{\text{eff}} = -\frac{2i|\mu_\perp \mu_\parallel|^{1/2}\mu_a}{\mu_a - i|\mu_\perp \mu_\parallel|^{1/2}}. \tag{12}$$

In integral equation (10), the quantity

$$S(n_x) = \lim_{\zeta \to 0} \left[\frac{2i\pi\chi}{Z_0 k_0} K_x(n_x, \zeta) - \ln \frac{k_0|\zeta|}{2} \right]$$

is determined by the regular part of the corresponding kernel, and is calculated numerically.

It can be shown that the solutions of integral equations (10) and (11) are the main terms of the asymptotics of exact solutions to initial integral equations (7) and (8) under conditions (9). It is a straightforward matter to verify that the solution of integral equation (10) with the logarithmic kernel automatically satisfies integral equation (11) with the Cauchy kernel [3]. This fact allows us to consider only equation (10). The solution of this equation can be found using the techniques discussed in [3] and has the form

$$\mathscr{I}(n_x, z) = \frac{2i}{Z_0 k_0 \sqrt{d^2 - z^2}} \frac{\chi \mathscr{E}_x^{\text{ext}}(n_x)}{\ln(4/k_0 d) - S(n_x)}. \tag{13}$$

Substituting the quantity $\mathscr{I}(n_x, z)$ from (13) into (4), we obtain the surface current density $I(x, z)$ of the strip antenna. The total current $I_\Sigma(x)$ in the cross section $x = \text{const}$ of the strip is determined by integrating the surface current density $I(x, z)$ over z between $-d$ and d:

$$I_\Sigma(x) = \frac{iV_0}{Z_0} \int_{-\infty}^{\infty} \frac{\sin(k_0 n_x b)}{k_0 n_x b} \frac{\chi \exp(-ik_0 n_x x)}{\ln(4/k_0 d) - S(n_x)} dn_x. \tag{14}$$

Note that in the general case, the integration over n_x in (14) can be performed only numerically. A closed-form expression for the current distribution can be derived if the strip is so narrow that the inequality $\ln(4/k_0 d) \gg |S(n_x)|$ is fulfilled for $|n_x| < (k_0 b)^{-1}$. In turn, the contribution from the greater values of $|n_x|$ to the current can be neglected under conditions (9). In this approximation, for $|x| > b$ we have

$$I_\Sigma(x) = \frac{V_0}{Z_0} \left(\frac{\varepsilon_{\text{eff}}}{\mu_{\text{eff}}} \right)^{1/2} \frac{\pi}{\ln(4/k_0 d)} \exp(-ih|x|), \tag{15}$$

where $h = k_0(\varepsilon_{\text{eff}}\mu_{\text{eff}})^{1/2}$, $\operatorname{Im} h < 0$, and $|h|b \ll 1$. It is evident that Eq. (15) corresponds to the transmission line theory with the complex current propagation constant h. Upon calculation of $I_\Sigma(x)$, the input impedance $Z = R + iX$ of the antenna can be found in a standard way as $Z = V_0/I_\Sigma(b)$ and, within the framework of approximation (15) under the condition $|h|b \ll 1$, is given by the formula

$$Z = Z_0(\mu_{\text{eff}}/\varepsilon_{\text{eff}})^{1/2} \pi^{-1} \ln(4/k_0 d).$$

It follows from (15) that in the case considered, it is inexpedient to increase the antenna length to values essentially exceeding the characteristic scale $|h|^{-1}$ of the current decrease along the antenna, because the real part of the antenna input impedance (and hence the power radiated by the antenna) stops to increase with the antenna length. It is also easy to show that the results obtained for an infinitely long strip can be generalized to the case of a finite-length antenna considered as a symmetric transmission line of length $2L$. To this end, the current can approximately be represented as

$$I_\Sigma(x) = I_0 \Big[\exp(-ih|x|) + R_1 \exp(-ihx) + R_2 \exp(ihx) \Big],$$

where $|x| < L$, I_0 is a certain constant, and h is a current propagation constant for an infinitely long antenna. Coefficients R_1 and R_2 can be determined from the condition that the current is zero at the antenna ends $x = \pm L$. As a result, the current distribution takes the form

$$I_\Sigma(x) = \frac{I_\Sigma(0)}{\sin(hL)} \sin[h(L - |x|)], \ |x| < L, \tag{16}$$

where $I_\Sigma(0)$ is the current at the antenna input. In the case of an electrically short antenna, which corresponds to the condition $|h|L \ll 1$, we arrive from (16) at the triangular shape of the current distribution along the antenna wire ($|x| < L$):

$$I_\Sigma(x) = I_\Sigma(0)(1 - |x|/L).$$

In the opposite case of a long antenna, current distribution (16) transforms (with allowance for the inequality $\operatorname{Im} h < 0$) to an exponential distribution described by formula (15).

The input impedance Z of a finite-length antenna with the known shape of the current distribution can be determined in a standard way using Poynting's theorem:

$$Z = -\int \hat{\mathbf{J}}^*(\mathbf{r}) \cdot \hat{\mathbf{E}}(\mathbf{r})\, d\mathbf{r}, \tag{17}$$

where $\hat{\mathbf{J}} = \mathbf{J}/I_\Sigma(0)$ is the density of the antenna current normalized to $I_\Sigma(0)$, $\hat{\mathbf{E}}$ is the field excited by this normalized current, and the asterisk denotes complex conjugation. Note that the value of the integral in (17) is determined by the form of the current and does not depend on $I_\Sigma(0)$.

5 Numerical Results

We now proceed to some numerical results illustrating the behavior of the current distribution of the considered strip antenna in the case where the isotropic medium is free space, i.e., $\varepsilon_a = 1$ and $\mu_a = 1$, whereas the hyperbolic metamaterial has the following parameters: $\varepsilon_\perp = 1$, $\varepsilon_\parallel = -2.15$, and $\mu_\perp = -\mu_\parallel = 1$. The computations were performed for the angular frequency $\omega = 10^9\ \text{s}^{-1}$ and the strip half-width

$d = 0.01$ cm. The midpoint of the interval to which the given voltage is supplied has the coordinate $x = 0$.

The behavior of the current magnitude $|I_\Sigma(x)|$ as well as the quantities $\mathrm{Re}\, I_\Sigma(x)$ and $\mathrm{Im}\, I_\Sigma(x)$, normalized to their values at the point $x = 0$, is illustrated by Figs. 2 and 3 under the condition $|h|b \ll 1$. For the chosen parameters, the results of calculations by the rigorous formula (14) and the approximate expression (15) coincide with graphical accuracy. In addition, the dashed lines in Figs. 2 and 3 show the analogous dependences obtained in the case where the hyperbolic medium has the following parameters: the permittivity tensor elements are still equal to $\varepsilon_\perp = 1$ and $\varepsilon_\parallel = -2.15$, while the elements of the permeability tensor have identical signs and are equal to $\mu_\perp = \mu_\parallel = 1$. In this case, one

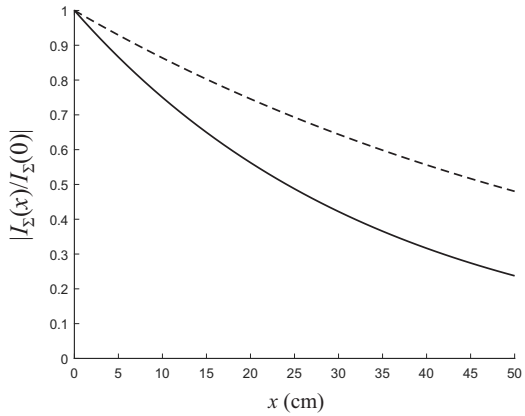

Figure 2. Normalized magnitude of the antenna current as a function of the x coordinate for $\mu_\perp = -\mu_\parallel = 1$ (solid line) and $\mu_\perp = \mu_\parallel = 1$ (dashed line) if $\varepsilon_a = \mu_a = 1$, $\varepsilon_\perp = 1$, $\varepsilon_\parallel = -2.15$, $\omega = 10^9$ s^{-1}, and $d = 0.01$ cm.

should use the above-derived expression for $\varepsilon_{\mathrm{eff}}$, but put $\mu_{\mathrm{eff}} = 1$ throughout. This fact can be established by making derivations similar to those in the preceding sections, but performed for $\mu_\perp = \mu_\parallel = 1$. Here, we do not dwell on such analysis for brevity. It is interesting to mention, that analogous results can also be obtained if the anisotropic medium on the one side of the interface is replaced by a strongly magnetized plasma. As is known, the normalized (to μ_0) magnetic permeability of a plasma medium is a scalar quantity equal to unity and, hence, $\mu_{\mathrm{eff}} = 1$. At the same time, the dielectric permittivity tensor of a strongly magnetized plasma becomes diagonal. In the resonant frequency ranges [2], relation (1) is fulfilled for the elements of the plasma permittivity tensor. As a result, the influence of such a plasma medium on the antenna characteristics turns out to be similar to that of a hyperbolic dielectric metamaterial.

6 Conclusion

We have considered the problem of determining the current distribution and input impedance of a perfectly conducting strip antenna located on the interface between an isotropic medium and a uniaxial anisotropic metamaterial. The main

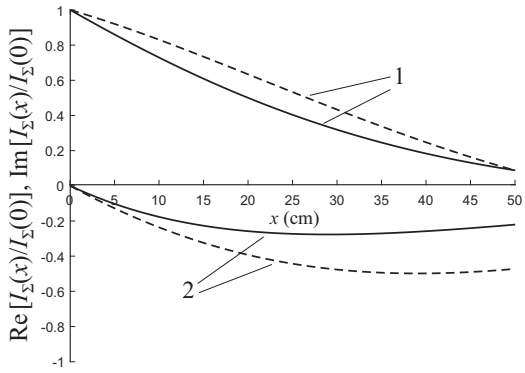

Figure 3. Normalized real (1) and imaginary (2) parts of the antenna current as functions of the x coordinate for $\mu_\perp = -\mu_\parallel = 1$ (solid lines) and $\mu_\perp = \mu_\parallel = 1$ (dashed lines). Other parameters are the same as in Fig. 2.

attention has been focused on the case where the metamaterial is hyperbolic. The problem has been reduced to solving a system of singular integral equations. On the basis of the solution of these equations, expressions for the current distribution and input impedance of the antenna have been obtained. It is shown that in the considered case, the properties of the integral equations admit a closed-form solution for the antenna current if the strip is sufficiently narrow. The derived solution, which describes the current distribution both along and across the strip, makes it possible to determine other electrodynamic characteristics of the considered strip antenna. It is shown that the results obtained for an infinitely long strip can be extended to a more realistic case of a finite-length antenna of the same configuration.

7 Acknowledgements

This work was supported by the Russian Science Foundation (project No. 14–12–00510).

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