32nd URSI GASS, Montreal, 19–26 August 2017

Guaranteed a Posteriori Estimates of Right-Hand Sides in Transmission Problems for Helmholtz Equations

Alexander G. Nakonechny(1), Yuri K. Podlipenko(1), and Yury V. Shestopalov(2)
(1) Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
(2) University of Gävle, Gävle, Sweden

Abstract

In this work we describe a method of obtaining guaranteed a posteriori estimates of unknown right-hand sides of the Helmholtz transmission problems from indirect measurements of a solution to this problem. The obtained results can be applied in various models of electromagnetics and acoustics that describe excitation of transparent bodies by sources of different kinds.

1 Introduction

In active location of objects, the problems arise of the determination of parameters of the wave fields from certain measurement results. Depending on information about characteristics of radiators and errors of measurements, different approaches of investigating such problems are used. In the case of random errors of measurements, methods of statistical analysis of random fields are employed [2, 3]. The presence of only deterministic errors leads to the necessity of applying novel methods.

To solve the problem of determining parameters of stationary wave fields, we use the methods of a posteriori estimation of parameters. Taking into account a priori information on unknown parameters and errors of measurements, we determine the set of such parameters that correspond to measurement data. Within this set, we choose the parameters optimal with respect to certain criteria. It is shown that such problems are associated with variational approaches to solve inverse problems.

A mathematical theory of guaranteed estimation developed in [4]–[8] creates foundation of the techniques for finding guaranteed a posteriori estimates for Helmholtz transmission problems with inexact data.

The methods further developed in this study are aimed at obtaining the most universal approach applicable for different types of excitations and measurement setting in scalar electromagnetic transmission problems.

2 Formulation of the problem

Let D be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ having the unit outward normal ν and $\varphi = (\varphi_1, \varphi_2)$ be a solution to the transmission problem

$$L_1\varphi_1 = B_1 f_1 \quad \text{in } D, \tag{1}$$

$$L_2\varphi_2 = B_2 f_2 \quad \text{in } \mathbb{R}^n \setminus D, \tag{2}$$

$$\mu_2\varphi_2 - \mu_1\varphi_1 = g_1 \quad \text{on } \Gamma, \tag{3}$$

$$\frac{\partial\varphi_2}{\partial\nu} - \frac{\partial\varphi_1}{\partial\nu} = g_2 \quad \text{on } \Gamma, \tag{4}$$

$$\frac{\partial\varphi_2}{\partial r} - ik\varphi_2 = o(1/r^{(n-1)/2}), \quad r = |x|,\ r \to \infty, \tag{5}$$

in which

- k_1, k_2, μ_1, and μ_2 are given nonzero complex numbers, $0 \le \arg k_s < \pi$ $(s = 1,2)$,
- $L_s\varphi_s = -(\Delta + k_s^2)\varphi_s$, $\partial\varphi_s/\partial\nu$ is a normal derivative of φ_s on the boundary Γ, $s = 1,2$,
- f_1 and f_2 are vectors from Hilbert spaces H_1 and H_2, respectively,
- $B_1 : H_1 \to L_2(D)$ and $B_2 : H_2 \to L_2(D_0)$ are bounded linear operators, where $L_2(D_0)$ is the space of all complex-valued functions that are square-integrable in a bounded domain D_0 and extended by zero on $\mathbb{R}^n \setminus D$,
- g_1 and g_2 are functions defined on Γ and belonging to Sobolev spaces $H^{3/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, correspondingly.

It can be shown that under the assumptions imposed on k_s and μ_s $(s = 1,2)$ in [1], the transmission problem (1)–(5) has a unique generalized solution such that $\varphi_1 \in H^2(D)$ and $\varphi_2 \in H^2_{\text{loc}}(\mathbb{R}^n \setminus D)$.

We suppose that the vector of measurements $y = (y_1, y_2) \in H_3 \times H_4$ corresponding to some unknown f_1, f_2, g_1, g_2 is given, where

$$y_s = C_s\varphi_s + v_s, \quad s = 1,2,$$

C_1, and C_2 are linear continuous operators that specify the method of measuring, v_1 and v_2 are unknown measurement errors, and (φ_1, φ_2) solves the problem (1)–(5).

Assume also that $(f_1, f_2, g_1, g_2, v_1, v_2) \in G$, where G is a given set in the space $\bar{H} \times H_3 \times H_4$, where

$$\bar{H} := H_1 \times H_2 \times H^{3/2}(\Gamma) \times H^{1/2}(\Gamma).$$

Denote by G_y the set

$$\big\{(f_1, f_2, g_1, g_2) : (f_1, f_2, g_1, g_2, y_1 - C_1\varphi_1, y_2 - C_2\varphi_2) \in G\big\}.$$

Let L be a continuous linear operator mapping the space $\bar{H}$ in some Hilbert space F and let $f = (f_1, f_2, g_1, g_2) \in \bar{H}$. Denote by $G_y(L)$ the set of all vectors of the form $\psi = Lf$ such that $f \in G_y$.

Guaranteed a posteriori estimate $\hat{\psi}$ of vector $\psi = Lf$ is found from the condition

$$\inf_{u \in G_y(L)} \sup_{\psi \in G_y(L)} \|u - \psi\|_F = \sup_{\psi \in G_y(L)} \|\hat{\psi} - \psi\|_F =: \rho_a.$$

The quantity ρ_a is called guaranteed error of a posteriori estimation. For various types of measurement operators C_1 and C_2 and for special sets G, including the case of an unbounded set G, we obtain representations for the guaranteed a posteriori estimates and guaranteed errors of a posteriori estimation. We also investigate the connection between such estimates and a priori estimates obtained in [4]–[6].

3 Measurement Examples

We consider two examples of measurements corresponding to different realizations of linear continuous operators specifying the measuring technique. The first one is

$$y_s(x) = (y_1^{(s)}(x), \ldots, y_{m_s}^{(s)}(x)), \quad s = 1, 2,$$

in which

- $y_k^{(1)}(x) = \int_{\Omega_k^{(1)}} h_k^{(1)}(x,y)\varphi_1(y)\,dy + v_k^{(1)}(x)$, $x \in \Omega_k^{(1)}$,
  $y_k^{(2)}(x) = \int_{\Omega_k^{(2)}} h_k^{(2)}(x,y)\varphi_2(y)\,dy + v_k^{(2)}(x)$, $x \in \Omega_k^{(2)}$,
- $\Omega_k^{(1)} \subset D$ and $\Omega_k^{(2)} \subset \mathbb{R}^n \setminus \bar{D}$ are nonintersecting bounded subdomains such that $\bar{\Omega}_k^{(1)} \subset D$ and $\bar{\Omega}_k^{(2)} \subset \mathbb{R}^n \setminus D$,
- kernels $h_k^{(1)} \in L^2(\Omega_k^{(1)} \times \Omega_k^{(1)})$ and $h_k^{(2)} \in L^2(\Omega_k^{(2)} \times \Omega_k^{(2)})$ are prescribed functions, and
- $v_k^{(s)}(x)$ are the errors of measurements.

The second example of measurement that we will study is

$$y_s = (y_1^{(s)}, \ldots, y_{m_s}^{(s)}), \quad s = 1, 2,$$

in which $y_k^{(s)} = \varphi_s(x_k^{(s)}) + v_k^{(s)}$, $x_k^{(1)} \in D$, $x_k^{(2)} \in \mathbb{R}^n \setminus D$, where $v_k^{(1,2)}$ are the measurement errors and $x_k^{(1,2)}$ may be referred to as the measurement point sets.

These measurement examples can find direct applications in scalar electromagnetic transmission problems by using appropriate definitions of the kernel functions and choice of the measurement point sets.

4 Conclusion

Beyond purely theoretical interest and in view of a universal character of the proposed settings for estimation problems and measurement methods, the obtained results can be used for creating mathematical models in automatized measurement data processing systems and used for interpretation of electromagnetic and acoustic observations with uncertain and noisy data.

5 Acknowledgements

The authors acknowledge support of The Swedish Institute, project Largescale, and University of Gävle, Sweden.

References

[1] R. Kress and G. F. Roach, "Transmission Problems for the Helmholtz Equation," *Journal of Mathematical Physics*, **19**, 1978, pp. 1433–1437, doi:10.1063/1.523808.

[2] A.V. Ivanov and N.N. Leonenko, "Statistical Analysis of Random Fields", Kluwer, 1989.

[3] A. Ramm, "Random Fields Estimation", World Scientific, 2005.

[4] A. Nakonechny, Y. Podlipenko, and Y. Shestopalov, "Guaranteed Estimation for Inverse Problems in Electromagnetics and Acoustics", *Proc. 18th International Conference on Electromagnetics in Advanced Applications*, ICEAA 2016, pp. 374–377, doi:10.1109/ICEAA.2016.7731403.

[5] Y. K. Podlipenko, A. G. Nakonechny, and Y. V. Shestopalov, "Guaranteed Estimation of Solutions to Helmholtz Problems from Pointwise Noisy observations", *Proc. 2016 International Conference Days on Diffraction*, pp. 336–339, doi:10.1109/DD.2016.7756869.

[6] Y. K. Podlipenko and Y. V. Shestopalov, "Guaranteed Estimation of Solutions to Transmission Problems for Helmholtz Equation with Uncertain Data From Their Indirect Noisy Observations", *Proc. 2016 URSI International Symposium on Electromagnetic Theory (EMTS)*, pp. 93–95, doi: 10.1109/URSI-EMTS.2016.7571321

[7] O.G. Nakonechny, "Optimal Control and Estimation in Partial Differential Equations", Kyiv State University, Kyiv, 2004.

[8] Y. Podlipenko and Y. Shestopalov, "Mixed Variational Approach to Finding Guaranteed Estimates for Solutions and Right-Hand Sides of the Second-Order Linear Elliptic Equations under Incomplete Data", *Minimax Theory Appl.*, **1**, 2, 2016, pp. 197–244.