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Methods for Verifying Solvability of the Permittivity Reconstruction in Canonical Waveguide Inverse Problems

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Abstract

We present a summary of mathematical methods that enable one to study and reveal the properties of the transmission coefficient as a function of problem parameters aimed at solution to some canonical waveguide inverse problems. The results are based on application of the theory of functions of several complex variables and singularities of differentiable mappings and lead to substantial improvement of the available techniques for permittivity reconstruction of material samples in waveguides.

1. Introduction

Determination of permittivity of layered dielectrics placed in a waveguide of rectangular cross section is known as a fundamental waveguide inverse problem. This setting has become canonical after the discovery of explicit formulas for calculating permittivity of a sample in the form of a single (one-layer) parallel-plane section of homogeneous dielectric medium using the so-called Nicolson-Ross-Weir (NRW) method [1]. The NRW method utilizes closed-form expressions for the material parameters determined from the measured S-parameters (transmission coefficient) and has been used since the beginning of 1970s as a standard technique for measuring the permittivity and permeability of homogeneous isotropic materials [1, 2]; this method is used in standard vector network analyzers (VNA) [3]. A broad analysis of the NRW approach is given in a recent work [4]. More detailed mathematical studies and applications of the closed-form reconstruction of permittivity of a one-layer parallel-plane section may be found in [5, 6]. Paper [7] presents comprehensive analysis of explicit expressions for multi-layer parallel-plane dielectric media and reports the occurrence of singularities of the function describing the transmission coefficient. In view of this fact, which is actually only one of the newly discovered reasons to reconsider NRW method (more reasons are discussed in this paper), application of the NRW-based technique requires deeper investigations and justification. This will be an objective of this work and a series of extended articles to be published as continuation of this study.

2. Mathematical Background

According to [5, 6], the transmission coefficient F in the problem of scattering of a principal waveguide mode by one or several (n) parallel-plane homogeneous dielectric layers placed in a standard single-mode waveguide of rectangular cross section can be obtained explicitly. Next, F is considered as a function of all $2n + 3$ problem parameters and reconstructing permittivity of one or several layers (sections) reduces to the equation

$$F(\bar{d}) = F^*, \qquad (1)$$
$$\bar{d} = (\bar{d}_1, \bar{d}_2), \bar{d}_1 = (f, a, b, \bar{l}_{(n)}), \bar{d}_2 = \bar{\varepsilon}_{(n)},$$

where f is the field frequency, a and b are the waveguide cross-sectional dimensions, $\bar{l}_{(n)}$ and $\bar{\varepsilon}_{(n)}$ are n-dimensional vectors of, respectively, the coordinates of n layers and their permittivities, and F^* is the given (measured) complex value of the transmission coefficient. There are two general settings associated with the inverse problem under study formulated in terms of equation (1): (a) determination of $\bar{\varepsilon}_{(n)} = \bar{\varepsilon}_{(n)}(\bar{d}_1, F^*)$ as a generalized n-dimensional hyper-surface in the $(2n + 4)$-dimensional parameter space defined implicitly by (1) and (b) analytical or numerical solution of (1) with respect to a subset of the $\bar{\varepsilon}_{(n)}$-components; the latter requires (local) inversion of function $F(\bar{d})$. There may be several simplified statements of this inverse problem used in practical calculations, in particular, the simplest versions: to determine the permittivity $\varepsilon_1 = \varepsilon_1(F^*)$ of a one-layer section of the width l_1 from the equation $F(\varepsilon_1) = F^*$, or $\varepsilon_1 = \varepsilon_1(f, F^*)$ or $\varepsilon_1 = \varepsilon_1(l_1, F^*)$ for all other parameters being fixed as implicit functions specified by (1). The proofs of existence of these implicitly defined functions and multi-dimensional hyper-surfaces require (i) determination of the *domain* D_F of function F, in other words, the sets of its singularities and extrema of all kinds and the subsets of D_F and the corresponding ranges where F is a one-to-one mapping, and (ii) formulation, analysis, and verification of the solvability conditions for equation (1) (including the existence of the inverse functions involved in the analysis). These problems still remain, to a big extent, unsolved; the corresponding proofs must be based on complete mathematical investigation of the properties of $F(\bar{d})$ as a function of several complex variables. In the absence of these mathematical results, the NRW method and similar approaches constituting a particular case of formulation of the canonical inverse problem in terms of the equation $F(\varepsilon_1) = F^*$ has no mathematical background and cannot be considered as a justified approach. Indeed, already in the simplest case of one layer in a waveguide of rectangular cross section the transmission coefficient has singularities and its derivatives zeros which *must* be taken into account when applying the explicit NRW-type solution formulas for determining the layer permittivity. Namely: considering, in view of [5—7], $F = F(z)$, $z = (\varepsilon_1 - \pi^2/(k_0 a)^2)^{1/2}$, where $k_0 = 2\pi f/c$ is the free-space wavenumber (c is the speed of light in vacuum) with all other parameters being fixed we have

$$F(z) = \frac{\exp(ist)}{\cos tz + iZ(z,s)\sin tz},$$

$$Z(z,s) = \frac{1}{2}\left(\frac{z}{s} + \frac{s}{z}\right), t = k_0 l_1,$$
$$s = (1 - \pi^2/(k_0 a)^2)^{1/2}.$$

One can show [7] that (i) $F(z)$ is a meromorphic function with an infinite countable set of complex poles $S_0 = \{\zeta_m{}^{\pm}, m = 1,2,\dots\}$ consisting of two components situated in the first and third quadrants in the complex z-plane; (ii) $F(z)$ has no zeros and no real singularities and is (locally) invertible in a certain band in the complex z-plane containing the real positive semi-axis; (iii) $|F(z)| \leq 1$ for real z and has (real) extrema at the points $z = z_m{}^* = \frac{\pi m}{t}$, so that $F'(z_m{}^*) = 0$ and the magnitude $|F(z)|$ attains its extreme values $|F(z_m{}^*)| = 1$ $(m = 1,2,\dots)$. In addition to this, the derivative $F'(z)$ has an infinite countable set of complex zeros in the first and third quadrants in the complex z-plane. Note that singularities and extrema of $F(\varepsilon_1)$ are actually three-dimensional hyper-surfaces in the parameter space (f, a, l_1).

3. Solvability of Canonical Inverse Problems

(i)—(iii) constitute only a part of important properties of the transmission coefficient that govern correct mathematical formulation and unique solvability of the canonical inverse problem of reconstructing (complex) permittivity of a one-layer homogeneous dielectric insert. These properties are preserved for arbitrary number of layers forming an inclusion in the waveguide in the sense that every layer 'generates' its own sequences of singularities and extrema points of $F = F(\bar{d})$ so that the transmission coefficient acquires the whole sets of singularities and extrema that form families of multi-dimensional hyper-surfaces. Knowledge of these sets is a *necessary* condition for validating unique solvability of the inverse problems under study and producing explicit closed-form expressions and numerical methods for calculating material parameters.

Analysis of the location of extremal and singular sets associated with function $F(\bar{d})$ employs parameter-continuation and parameter-differentiation techniques [7] which prove to be efficient for multi-layer parallel-plane dielectric inclusions: proceeding from the simplest case of a one layer where all necessary proofs and explicit formulas are available, one constructs e.g. permittivity as implicit function of one or several problem parameters using a reduction to the Cauchy problem for an ordinary or partial differential equations; the unique solvability of the obtained initial-value problems is validated using the first- or higher-order Cauchy-Kowalewski theorem [7, 8].

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5. Conclusion

The NRW method and similar techniques for solving inverse waveguide problems require complete mathematical justification including detailed investigation of the properties of the transmission coefficient and elaboration of a specific mathematical approach employing theory of functions of several complex variables and differential mappings.

6. References

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