Poincaré wavelets and their applications to Gaussian beam summation

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Abstract

The decomposition of solutions of an initial-boundary value problem for the wave equation in terms of localized solutions is presented. The decomposition is constructed with the mathematical technique of the continuous affine Poincaré analysis. In the case of two spatial dimensions, the obtained results are given as a fourfold integrals in the space of parameters of localized solutions. We apply this formula to the Helmholtz equation. As an example, we obtain a decomposition of the Green's function for the Helmholtz equation in terms of Gaussian beams. The results are in good agreement with a known formula of the Gaussian beam summation due to Popov.

1 Introduction

Representations of solutions of the wave equation or the Helmholtz equation in terms of its localized solutions with known properties are useful for some problems of diffraction and wave propagation [1, 2, 3], and, in particular, for inverse problems [4]. Decompositions in Gaussian beams was found in papers of [5, 6, 7] by asymptotic methods. Our results are based on methods of continuous wavelet analysis. Decompositions of solutions of the wave equation with constant speed in terms of localized solutions obtained by the similitude group were presented in [8, 9, 10]. We give here decompositions based on the affine Poincaré group. For the first time, the decomposition of an initial-boundary value problem for the wave equation in a half-plane was suggested in our papers [11, 12, 13]. The medium of the half-plane was assumed to be homogeneous. The case of an inhomogeneous half-plane and its application to the inverse problem of seismics was treated in [14]. It was shown in [14] by numerical calculations that although the decomposition has the form of a multiple integral, the obtained formulas are efficient, because the integrand is very sparse.

Here we discuss the application of the decomposition based on affine Poincaré wavelets to the decomposition in monochromatic Gaussian beams, i.e., solutions of the wave equation localized near a line. We give our considerations for a homogeneous medium and obtain a formula of Gaussian beam summation [6] as a result of an asymptotic evaluation by the stationary phase method of the wavelet-based integral representation of the Green's function.

2 Wavelet-based decompositions of solutions of the wave equation

We deal here with solutions of the wave equation

$$\Delta u(t,x,z) - \frac{1}{c^2}\frac{\partial^2 u(t,x,z)}{\partial t^2} = 0, \quad z \geq 0,$$
$$u\,|_{z=0} = f(t,x);$$

with some causality condition, i.e., no waves come from infinity. We study solutions of the form

$$u(t,x,z) = \int d\mu(\nu)\mathscr{F}(\nu)\Psi_\nu(t,x,z),$$

where ν is a set of parameters, $d\mu(\nu)$ is a measure in the space of parameters, $\Psi_\nu(t,x,z)$ is a set of solutions of the homogeneous wave equation parameterized by ν, which are called elementary and have known properties. The excitation coefficient for the particular elementary solution $\mathscr{F}(\nu)$ depends on the boundary data $f(t,x)$. The boundary condition

$$u(t,x,z)\,|_{z=0} = f(t,x) = \int d\mu(\nu)\mathscr{F}(\nu)\psi_\nu(t,x),$$

where $\psi_\nu(t,x) = \Psi_\nu(t,x,0)$ should be satisfied.

Generally speaking, the solution $u(t,x,z)$ can be decomposed into two terms; one of them is a solution propagating away from the boundary and the other is a decaying solution. We are interested here only in the solution that propagates. To obtain only a propagating solution, we assume that $\hat{f}(\omega,k_x) = 0$ if $|k_x| > \omega/c$, i.e., we require that the boundary data can be represented as the following Fourier integral:

$$f(t,x) = \frac{1}{(2\pi)^2}\int_0^\infty d\omega \int_{-\omega/c}^{\omega/c} dk_x \hat{f}(\omega,k_x)e^{-i\omega t + ik_x x}. \quad (1)$$

Here we study the representation of the solution obtained by us in [13] for a constant speed of wave propagation, which satisfies the causality condition that all Fourier components of the solution satisfy the limiting absorption principle. It has a form of a multiple improper integral

$$u(t,x,z) = \frac{1}{C_{\zeta\psi}}\int_{-\pi/2}^{\pi/2}\frac{d\theta}{\cos\theta}\int_0^\infty \frac{da}{a^3}\iint_{R^2} dt_s dx_s \quad (2)$$
$$F_{\theta a}(t_s,x_s)\Psi_{\theta a}(t-t_s,x-x_s,z).$$

To determine $\Psi_{\theta a}(t-t_s,x-x_s,z)$, we should choose some one solution of the wave equation $\Psi(t,x,z)$ satisfying the causality condition. The function $\psi(t,x)=\Psi(t,x,0)$ is named *the mother wavelet*. A family of the affine Poincaré wavelets, $\psi_{\theta a}(t,x)$, is constructed from $\psi(t,x)$ by transformations of the Poincaré group with scaling, i.e.,

$$\psi_{\theta a}(t,x)=\frac{1}{a}\psi\left(\frac{t'}{a},\frac{x'}{a}\right), \tag{3}$$

$$\begin{pmatrix} ct' \\ x' \end{pmatrix}=\Lambda_{-\phi}\begin{pmatrix} ct \\ x \end{pmatrix}, \tag{4}$$

$$\Lambda_\phi=\begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}, \tag{5}$$

where the parameter c is the wave velocity. Solutions from the family $\Psi_{\theta a}(t-t_s,x-x_s,z)$, which we call *elementary solutions*, satisfy the boundary condition

$$\Psi_{\theta a}(t-t_s,x-x_s,0)=\psi_{\theta a}(t-t_s,x-x_s) \tag{6}$$

and the causality condition. The medium is homogeneous, therefore

$$\Psi_{\theta a}(t-t_s,x-x_s,z)=\frac{1}{a}\Psi\left(\frac{t'}{a},\frac{x'}{a},\frac{z}{a}\right). \tag{7}$$

The parameters x_s and t_s characterize shifts in the x coordinate and in time t, respectively; a rules a scaling of the solution. The matrix Λ_ϕ determines the Lorentz boost (or hyperbolic rotation), which is characterized by the parameter ϕ. We use in (3) an angle θ, which is linked with ϕ as follows:

$$\tan\theta=\sinh\phi; \tag{8}$$

Here θ is introduced formally. For some particular choice of $\Psi(t,x,z)$, which will be discussed later, the parameter θ has the meaning of the angle of the exit of the solution from the boundary. The improper integral converges in a weak sense, or in the sense of distributions.

In general case the condition (1) is not fulfilled, the boundary data should be decomposed into the direct sum of the 4 components with Fourier transforms lying in domains: $\omega>0, ck_x<\omega$(discussed here); $\omega<0, ck_x<|\omega|$; $k_x>0, ck_x>|\omega|$; $k_x<0, c|k_x|>|\omega|$. These cases should be considered in the same manner, see details in [13].

We note that although relativistic concepts are used for construction of elementary solutions, the obtained results can be applied for nonrelativistic problems of wave propagation.

The excitation coefficient $F_{\theta a}(t_s,x_s)$ is determined as the affine Poincaré wavelet transform of the function $f(t,x)$,

$$F_{\theta a}(t_s,x_s)=\int\limits_{\mathbb{R}^2} dxdt\, f(t,x)\overline{\zeta_{\theta a}(t-t_s,x-x_s)}, \tag{9}$$

where the family $\zeta_{\theta a}(t,x)$ is constructed from a function $\zeta(t,x)$ called a mother wavelet as follows:

$$\zeta_{\theta a}(t,x)=\frac{1}{a}\zeta\left(\frac{t'}{a},\frac{x'}{a}\right), \tag{10}$$

where (t',x') are given by (4).

The mother wavelets $\psi(t,x)$ and $\zeta(t,x)$ must satisfy the admissibility condition,

$$0<C_{\zeta\psi}=\int\limits_0^\infty d\omega \int\limits_{|k_x|<\omega/c} dk_x \frac{\overline{\hat{\zeta}(\omega,k_x)}\hat{\psi}(\omega,k_x)}{\omega^2/c^2-k_x^2}<\infty; \tag{11}$$

here the notation $\hat{\zeta}(\omega,k_x)$, $\hat{\psi}(\omega,k_x)$ stand for the Fourier transform of $\zeta(t,x)$ and $\psi(t,x)$, respectively.

Here the formulas are valid if both mother wavelets are square integrable in variables t and x. However they can be applied also to distributions.

If we choose $\zeta(t,x)=\delta(t)\delta(x)$, we obtain

$$\zeta_{\theta a}(t,x)=a\delta(t)\delta(x), \quad F_{\theta a}(t_s,x_s)=af(t_s,x_s).$$

The decomposition (2) in this case reads

$$u(t,x,z)=\frac{1}{C_{\zeta\psi}}\int\limits_{-\pi/2}^{\pi/2}\frac{d\theta}{\cos\theta}\int\limits_0^\infty\frac{da}{a^2}\iint\limits_{R^2} dt_s dx_s \; f(t_s,x_s)\Psi_{\theta a}(t-t_s,x-x_s,z). \tag{12}$$

Instead of a and θ, we introduce new variables ω and θ in such a way that $\omega=1/(a\cos\theta)$. Then we obtain

$$u(t,x,z)=\frac{1}{C_{\zeta\psi}}\int\limits_{-\pi/2}^{\pi/2} d\theta\cos\theta\int\limits_0^\infty d\omega\omega\iint\limits_{R^2} dt_s dx_s \; f(t_s,x_s)\tilde{\Psi}(t-t_s,x-x_s,z;\omega,\theta), \tag{13}$$

$$\tilde{\Psi}(t-t_s,x-x_s,z;\omega,\theta)=\Psi(\frac{t'}{a},\frac{x'}{a},\frac{z}{a}),$$

$$a=1/(\omega\cos\theta).$$

3 Elementary solutions

Our aim is to give an example of the application of formula (2). We intend to obtain a decomposition of a solution of the wave equation in Gaussian beams, i.e., monochromatic solutions of the wave equation localized in the Gaussian manner near a ray. In a homogeneous medium, the ray is a straight line. The Gaussian beam can be constructed by means of the Green's function $G(x,z;\omega)$ for the Helmholtz equation

$$\left(\Delta+\frac{\omega^2}{c^2}\right)G(x,z;\omega)=\delta(\vec{r}), \quad \vec{r}=(x,z). \tag{14}$$

As an additional condition, we choose the limited absorption principle; therefore, $G(x,z;\omega)=-\frac{i}{4}H_0^{(1)}(kr)$, where $r=|\vec{r}|$, $H_0^{(1)}(kr)$ is the Hankel function, $k=\frac{\omega}{c}$. We shift the z coordinate in the complex plane, i.e., replace z by $z-i\varepsilon$, $\varepsilon=k\sigma_\perp^2$, take an asymptotics of the Hankel function, and expand r in powers of $x/(z-i\varepsilon)$. Finally, adding a factor $e^{-i\omega t}$, we get an asymptotic solution, called the monochromatic Gaussian beam

$$G(x,z-i\varepsilon;\omega) = \alpha B(x,z;\omega,\sigma_\perp),$$

$$B(x,z;\omega,\sigma_\perp) = \frac{\exp\left(ikz-\frac{x^2}{2(\sigma_\perp^2+\frac{iz}{k})}\right)}{\sqrt{1+\frac{iz}{k\sigma_\perp^2}}}, \quad (15)$$

$k=\frac{\omega}{c}$, $\alpha=\frac{i}{2\sqrt{2\pi}k\sigma_\perp}$. This solution is localized in the Gaussian manner near the line $x=0$. We take

$$\Psi(t,x,z) = e^{-i\omega t}B(x,z;\omega,\sigma_\perp) \quad (16)$$

with some values of parameters ω, $\sigma_\perp$.

4 Summation of Gaussian beams due to Popov

We intend to specify our results (2) by a choice of Gaussian beams as $\Psi(t,x,z)$. We compare results with the Gaussian beam summation formula suggested by Popov M.M., see [4]. It is based on the Kirchhoff integral and in our notation reads

$$u(t,x,z) \simeq -\frac{2}{\pi}\mathrm{Re}\int_{\mathbb{R}^2} dt_s dx_s \int_0^\infty d\omega\, e^{-i\omega(t-t_s)} f(t_s,x_s)\, \frac{\partial G(x-x_s,z-z_s;\omega)}{\partial z_s}\bigg|_{z_s=0}, \quad (17)$$

where the Green's function is asymptotically represented as the integral superposition of the Gaussian beams

$$\frac{\partial G(x,z-z_s;\omega)}{\partial z_s}\bigg|_{z_s=0} \simeq \frac{\omega}{4c^2}\int_0^{2\pi} d\theta \cos\theta B_\theta(x,z;\omega,\sigma_\perp), \quad (18)$$

where $B_\theta(x,z;\omega,\sigma_\perp)$ is the field of the Gaussian beam with axis inclined to ox, which forms the angle θ with the vertical. This Gaussian beam is convenient to give in the orthogonal coordinate system $(\tilde{x},\tilde{z})$:

$$\begin{pmatrix}\tilde{x}\\ \tilde{z}\end{pmatrix} = \boldsymbol{M}_\theta \begin{pmatrix}x\\ z\end{pmatrix},\ \boldsymbol{M}_\theta = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}. \quad (19)$$

We have denoted

$$B_\theta(x,z;\omega,\sigma_\perp) \equiv B(\tilde{x},\tilde{z};\omega,\tilde{\sigma_\perp}). \quad (20)$$

Therefore θ is the exit angle of the beam from a point source. After substitution of (18) into (17), we obtain the Gaussian beam summation formula [4]:

$$u(t,x,z) \simeq \frac{1}{2\pi^2}\mathrm{Re}\iint_{\mathbb{R}^2} dt_s dx_s \int_0^\infty d\omega\omega \int_{-\pi/2}^{\pi/2} d\theta\cos\theta\, f(t_s,x_s)\,\tilde{\Psi}(t-t_s,x-x_s,z;\omega,\theta). \quad (21)$$

$$\tilde{\Psi}(t,x,z;\omega,\theta) = e^{-i\omega t}B_\theta(x,z;\omega,\sigma_\perp) \quad (22)$$

5 Our approach to the decomposition of the derivative of Green's function

Our aim here is to obtain (18) from (2) by an appropriate choice of mother wavelets $\psi(t,x)$, $\zeta(t,x)$ and asymptotic simplification. To do this, we need to have $\frac{\partial G(x,z-z_s;\omega)}{\partial z_s}$ at the boundary, i.e., for $z=0$. Calculations in the space of distributions show

$$\frac{\partial G(x,z-z_s;\omega)}{\partial z_s}\bigg|_{z_s=0} \xrightarrow[z\to 0]{} \frac{1}{2}\delta(x). \quad (23)$$

Therefore, for the solution of the wave equation we have

$$f(ct,x) = \frac{1}{2}e^{-ikct}\delta(x). \quad (24)$$

Calculations by formulas (9), (10) yield

$$\mathscr{F}_{\phi a}(ct_s,x_s) = \frac{a}{2}\exp(-ikct_s)\,\delta(x_s). \quad (25)$$

We calculate a family of solutions by means of formulas (15) and (3), and substitute these solutions and (25) into (2). The integral in x_s can be easily taken, the integrals in t_s and a are calculated by the stationary phase method. The applicability conditions of the stationary phase method produces restrictions to parameters of the problem.

Finally, we obtain an integral in the parameter θ, which has the meaning of the angle between the axis of the beam and the vertical, i.e., the z direction.

$$\frac{\partial G(x,z-z_s;\omega)}{\partial z_s}\bigg|_{z_s=0} = \frac{\omega}{2c^2}\int_{-\pi/2}^{\pi/2} d\theta\cos\theta B_\theta(x,z;\omega,\tilde{\sigma_\perp}),$$

$\tilde{\sigma_\perp} = \sigma_\perp/\cos\theta$, where the integrand is the Gaussian beam propagated along the axis $\tilde{z}$, which has the parameter of width $\sigma_\perp$ dependent on the angle θ with the vertical. To obtain this formula, we neglect the term $\tilde{x}\tan\theta$ comparing with $k\tilde{\sigma}_\perp^2$, assuming that $k\sigma_\perp \gg 1$ and $\tan\theta$ is not too large.

6 Decompositions of solutions for inhomogeneous media

The decomposition of solutions of the wave or Helmholtz equation in terms of Gaussian beams has applications to the inverse problem of seismics named the migration problem. The medium is assumed to be inhomogeneous, the smooth averaged speed of wave propagation is known. The aim is to find the discontinuity of the speed by using experimental data. The known algorithm reduces the migration problem to the solution of the initial-boundary value problem in a half-plane with a smoothly changing medium, see references in [14], [4]. Formula (2) was generalized in [14] to an inhomogeneous medium. The medium in the close proximity to the boundary of the medium was assumed to be homogeneous. The only distinction from (2) is the constructions of the family of solutions $\Psi_{\theta a}$. Instead of $\Psi_{\theta a}(t-t_s, x-x_s, z)$, one should write $\Psi_{\theta a}(t,x,z;t_s,x_s)$, because the solutions of the family cannot be calculated directly by applying transformations mentioned above to the solution $\Psi(t,x,z)$ in the inhomogeneous medium. The solutions must satisfy the boundary condition

$$\Psi_{\theta a}(t,x,0;t_s,x_s) = \psi_{\theta a}(t-t_s, x-x_s),$$

and the causality condition. We satisfy this condition approximately choosing the asymptotic solution $\Psi(t,x,z)$, which is called the quasiphoton (see discussion and references in [14]) and for which we can determine the direction of propagation. This solution is localized in the Gaussian manner near the point moving along the geometric optical ray issued from the boundary under the angle θ determined in (8). The important property of this solutions concerns the application to it of the affine Poincaré transformations. These tranformations map quasiphotons to quasiphotons with different parameters. Quasiphotons calculated on the boundary are converted to the Morlet wavelet the well-known in wavelet analysis. Quasiphotons were found numerically for each of the parameters. All the necessary formulas, and also results of their numerical implementation can be found in our paper [14]. The numerical calculations based on formulas with multiple integrals proved their efficiency.

7 Conclusions

We conclude that formulas (2), (12) are similar to formulas of Gaussian beam summation [6]. The decomposition of the normal derivative of the Green's function can be got from these formulas by a special choice of elementary solutions and mother wavelets. We also discuss briefly necessary modifications to obtain a solution of the initial-boundary value problem in an inhomogeneous half-plane and give references to our results of numerical implementation of these formulas.

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