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Power, Energy Density, and Group/Energy-Transport Velocities in Spatially Dispersive Media

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Abstract

Power and stored energy density in lossless, spatially dispersive, anisotropic media are determined from the boundary conditions across a free-space/media interface and the wavepacket group velocity expressed as the ratio of power to energy density, that is, as an energy-transport velocity.

1 Introduction

It is proven in [1] that three-dimensional (3D), cubic, periodic arrays satisfy the Maxwellian macroscopic continuum equations if the normalized free-space and medium electrical separation distances are small enough[1] that the unit-cell averages of the microscopic fields and polarizations are approximately equal to the corresponding field and polarization coefficients of the fundamental Floquet mode. On the other hand, it is proven in [2, 3] that for lossless, reciprocal spatially dispersive 3D periodic arrays, the unit-cell-averaged microscopic Poynting vector (assumed to be the local power per unit area) is equal to that of the fundamental Floquet mode, namely $\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*)/2$, plus an additional term that depends on the spatial dispersion, even if the electrical separation distances are sufficiently small for the array to well approximate a continuum. In other words, for spatially dispersive periodic continua, the higher-order Floquet modes can contribute significantly to the average power even though they contribute negligibly to the macroscopic fields and polarizations.

Since $\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*)\cdot\hat{\mathbf{n}}/2$ integrated over a closed (or planar) surface (with outward unit normal $\hat{\mathbf{n}}$) *in free space* is unequivocally equal to the average power leaving the volume (or half space) defined by that surface [4], it follows from the aforementioned theorems in [1] and [2, 3] that the tangential components of the $\mathbf{E}$ and $\mathbf{H}$ fields cannot, in general, be continuous across the interface between free space and a spatially dispersive macroscopic continuum. Thus, one of the purposes of this paper is to use the recently formulated approach in [5] to determine the boundary conditions across the interface between free space and a representative spatially dispersive medium, namely an electric quadrupolar continuum characterized by a lossless, reciprocal, spatially dispersive dyadic effective permittivity $\overline{\boldsymbol{\varepsilon}}(\mathbf{k},\omega)$ with real ω angular frequency and real $\mathbf{k}$ propagation vector. It is shown that the same power results from these boundary conditions as the power derived in [2,3] for the fundamental Floquet mode of a lossless, reciprocal, spatially dispersive periodic array.

After establishing that the expression for power in a periodic array holds also for a representative spatially dispersive continuum that is not a periodic array, we derive a power and energy density whose ratio (energy-transport velocity) equals the group velocity of a source-free wavepacket in a lossless, spatially dispersive, anisotropic continuum. This ratio has the correct expression for continuum power in its numerator and, thus, the denominator conveniently reveals the expression for the stored energy density in a lossless, spatially dispersive, anisotropic continuum. Landau and Lifshitz [6, sec. 103] and Agranovich and Ginzburg [7, ch. 2] use similar expressions for power and stored energy density in a lossless, spatially dispersive, dielectric continuum consistent with the particular group-velocity ratio mentioned above. However, since there may be more than one power and energy density whose ratio is equal to the group velocity, it appears that the derivation of the correct power and stored energy density in lossless spatially dispersive continua (as well as spatially nondispersive continua [4]) must rely on the result that the integral of the outward-normal Poynting vector over a planar surface in free space is the power leaving the half space defined by that surface; then relate the free-space Poynting vector to the power in the continuum by means of the boundary conditions across a free-space/continuum interface, as is done in the present paper.

2 Boundary Conditions for Spatially Dispersive Continua

Maxwell's continuum equations for $e^{-i\omega t}$ time dependence can be written as

$$\nabla\times\mathscr{E}(\mathbf{r}) - i\omega\mathscr{B}(\mathbf{r}) = 0 \tag{1a}$$

$$\nabla\times\mathscr{H}(\mathbf{r}) + i\omega\mathscr{D}(\mathbf{r}) = \mathscr{J}(\mathbf{r}) \tag{1b}$$

with constitutive relations

$$\mathscr{D}(\mathbf{r}) = \varepsilon_0\mathscr{E}(\mathbf{r}) + \mathscr{P}(\mathbf{r}) \tag{2a}$$

$$\mathscr{H}(\mathbf{r}) = \mathscr{B}(\mathbf{r})/\mu_0 - \mathscr{M}(\mathbf{r}) \tag{2b}$$

[1] Generally "small enough" requires $k_0 d \ll 1$ and $|\mathbf{k}d| \ll 1$, where d is the cubic lattice spacing and k_0 and $\mathbf{k}$ are the free space and medium propagation constants, respectively.

where the electric and magnetic polarization densities $(\mathscr{P},\mathscr{M})$ can contain dipoles and all higher-order multipoles [8, ch. 1], [9].

In order to work with a physically realizable and reasonable spatially dispersive polarization (having constitutive relations that depend on the spatial derivatives of electric or magnetic fields), consider the effective electric polarization density of an electric quadrupolar continuum, namely $\mathscr{P}(\mathbf{r}) = -\nabla\cdot\overline{\mathscr{Q}}(\mathbf{r})/2$ with the constitutive relation for the dyadic electric quadrupolarization density $\overline{\mathscr{Q}}(\mathbf{r})$ given by [5,11]

$$\overline{\mathscr{Q}}(\mathbf{r}) = \alpha_Q\varepsilon_0\left\{\frac{1}{2}[\nabla\mathscr{E}(\mathbf{r})+\mathscr{E}(\mathbf{r})\nabla]-\frac{1}{3}\nabla\cdot\mathscr{E}(\mathbf{r})\overline{\mathbf{I}}\right\} \quad (3)$$

where α_Q is the electric quadrupolarizability density (real-valued for lossless continua) and $\mathscr{E}(\mathbf{r})\nabla$ denotes the transpose of the dyadic $\nabla\mathscr{E}(\mathbf{r})$. For an interface between free space and the continuum at $z=0$, the constitutive equation in (3) holds everywhere if a unit step function $u(z)$ is included that increases the electric quadrupolarizability density from its value of zero in free space to its value of α_Q in the material over a thin transition layer $0\le z\le \ell$ [5]; that is

$$\overline{\mathscr{Q}}_0(\mathbf{r}) = u(z)\alpha_Q\varepsilon_0\left\{\frac{1}{2}[\nabla\mathscr{E}(\mathbf{r})+\mathscr{E}(\mathbf{r})\nabla]-\frac{1}{3}\nabla\cdot\mathscr{E}(\mathbf{r})\overline{\mathbf{I}}\right\} \quad (4)$$

where the subscript "0" indicates that this constitutive relation holds for all $\mathbf{r}$. Then the Maxwell equations (1) become for an electric quadrupolar half-space interfacing with a free-space half space

$$\nabla\times\mathscr{E}(\mathbf{r}) - i\omega\mu_0\mathscr{H}(\mathbf{r}) = 0 \quad (5a)$$

$$\nabla\times\mathscr{H}(\mathbf{r}) + i\omega\varepsilon_0\mathscr{E}(\mathbf{r}) - \frac{1}{2}i\omega\nabla\cdot\overline{\mathscr{Q}}_0(\mathbf{r}) = \mathscr{J}(\mathbf{r}). \quad (5b)$$

Equations (4) and (5) were solved in [5] to obtain the following boundary conditions across the free-space/quadrupolar-continuum interface with unit normal $\hat{\mathbf{n}}$

$$\mathscr{E}_s^{(2)} - \mathscr{E}_s^{(1)} = 0 \quad (6)$$

$$\mathscr{H}_s^{(2)} - \mathscr{H}_s^{(1)} = -\frac{i\omega}{2}\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\cdot\overline{\mathscr{Q}}^{(2)}) \quad (7)$$

$$\hat{\mathbf{n}}\cdot\overline{\mathscr{Q}}^{(2)}\cdot\hat{\mathbf{n}} = 0 \quad (8)$$

with the superscripts "(1)" and "(2)" referring to the free-space and electric quadrupolar continuum sides of the interface transition layer, respectively, and the subscript "s" referring to tangential components. Three boundary conditions (rather than two) are required because an evanescent mode as well as a propagating mode can be excited in the lossless quadrupolar half space. However, in the present paper, we will be dealing with only propagating modes in wavepackets and, thus, we require two boundary conditions across the interface for a propagating mode, that is, at a low enough frequency that the evanescent mode can be considered part of the transition layer; specifically, it can be shown that this requires $k^2\alpha_Q \ll 1$, where $\mathbf{k}$ is the real propagation vector in the lossless quadrupolar continuum. These two boundary conditions are also given in [5] as

$$\mathscr{E}_s^{(2)} - \mathscr{E}_s^{(1)} = \frac{1}{2\varepsilon_0}\nabla_s(\hat{\mathbf{n}}\cdot\overline{\mathscr{Q}}^{(2)}\cdot\hat{\mathbf{n}}) \quad (9a)$$

$$\mathscr{H}_s^{(2)} - \mathscr{H}_s^{(1)} = -\frac{i\omega}{2}\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\cdot\overline{\mathscr{Q}}^{(2)}). \quad (9b)$$

2.1 Propagating plane waves

For plane-wave excitation in a continuum, $\mathscr{J}(\mathbf{r})e^{-i\omega t} = \mathbf{J}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ and Maxwell's equations in (1) transform to

$$\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega) - \omega\mathbf{B}(\mathbf{k},\omega) = 0 \quad (10a)$$

$$\mathbf{k}\times\mathbf{H}(\mathbf{k},\omega) + \omega\mathbf{D}(\mathbf{k},\omega) = -i\mathbf{J}(\mathbf{k},\omega) \quad (10b)$$

For linear anisotropic media, the constitutive relations in (2) become

$$\mathbf{D}(\mathbf{k},\omega) = \overline{\boldsymbol{\varepsilon}}(\mathbf{k},\omega)\cdot\mathbf{E}(\mathbf{k},\omega) \quad (11a)$$

$$\mathbf{B}(\mathbf{k},\omega) = \overline{\boldsymbol{\mu}}(\mathbf{k},\omega)\cdot\mathbf{H}(\mathbf{k},\omega). \quad (11b)$$

The plane-wave current source $\mathbf{J}(\mathbf{k},\omega)$ in (10b) allows $\mathbf{k}$ and ω in (10)–(11) to vary independently. In a lossless spatially dispersive continuum, the permittivity and permeability dyadics obey the relations [1]

$$\overline{\boldsymbol{\varepsilon}}(\mathbf{k},\omega) = \overline{\boldsymbol{\varepsilon}}^{T*}(\mathbf{k},\omega),\ \ \overline{\boldsymbol{\mu}}(\mathbf{k},\omega) = \overline{\boldsymbol{\mu}}^{T*}(\mathbf{k},\omega) \quad (12)$$

and if the material is reciprocal [1]

$$\overline{\boldsymbol{\varepsilon}}(-\mathbf{k},\omega) = \overline{\boldsymbol{\varepsilon}}^{T}(\mathbf{k},\omega),\ \ \overline{\boldsymbol{\mu}}(-\mathbf{k},\omega) = \overline{\boldsymbol{\mu}}^{T}(\mathbf{k},\omega) \quad (13)$$

where the superscript "T" denotes the transpose of a dyadic. Upon substitution from (11), the equations in (10) become

$$\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega) - \omega\overline{\boldsymbol{\mu}}(\mathbf{k},\omega)\cdot\mathbf{H}(\mathbf{k},\omega) = 0 \quad (14a)$$

$$\mathbf{k}\times\mathbf{H}(\mathbf{k},\omega) + \omega\overline{\boldsymbol{\varepsilon}}(\mathbf{k},\omega)\cdot\mathbf{E}(\mathbf{k},\omega) = -i\mathbf{J}(\mathbf{k},\omega) \quad (14b)$$

which, for the electric quadrupolar continuum, can be re-expressed as

$$\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega) - \omega\mu_0\mathbf{H}(\mathbf{k},\omega) = 0 \quad (15a)$$

$$\mathbf{k}\times\mathbf{H}(\mathbf{k},\omega) + \omega\varepsilon_0[1+\overline{\boldsymbol{\chi}}_e(\mathbf{k},\omega)]\cdot\mathbf{E}(\mathbf{k},\omega) = -i\mathbf{J}(\mathbf{k},\omega) \quad (15b)$$

where $\overline{\boldsymbol{\varepsilon}}(\mathbf{k},\omega) = \varepsilon_0[1+\overline{\boldsymbol{\chi}}_e(\mathbf{k},\omega)]$ and the electric susceptibility dyadic $\overline{\boldsymbol{\chi}}_e(\mathbf{k},\omega)$ is obtained from (3) and (5) as

$$\overline{\boldsymbol{\chi}}_e(\mathbf{k},\omega) = \frac{\alpha_Q}{4}\left(k^2\overline{\mathbf{I}}+\frac{1}{3}\mathbf{k}\mathbf{k}\right). \quad (16)$$

For these plane wave fields, the boundary conditions in (9) become

$$\mathbf{E}_s^{(2)} - \mathbf{E}_s^{(1)} = \frac{1}{2\varepsilon_0}(\mathbf{k}-k_z\hat{\mathbf{z}})(\hat{\mathbf{z}}\cdot\overline{\mathbf{Q}}^{(2)}\cdot\hat{\mathbf{z}}) \quad (17a)$$

$$\mathbf{H}_s^{(2)} - \mathbf{H}_s^{(1)} = -\frac{i\omega}{2}\hat{\mathbf{z}}\times(\hat{\mathbf{z}}\cdot\overline{\mathbf{Q}}^{(2)}) \quad (17b)$$

with

$$\overline{\mathbf{Q}}^{(2)} = i\alpha_Q\varepsilon_0\left[\frac{1}{2}(\mathbf{k}\mathbf{E}^{(2)}+\mathbf{E}^{(2)}\mathbf{k})-\frac{1}{3}(\mathbf{k}\cdot\mathbf{E}^{(2)})\bar{\mathbf{I}}\right]. \quad (18)$$

For source-free fields in the lossless quadrupolar half space, the propagating modes are transverse electromagnetic (TEM) and, thus, for the source-free propagating modes $\mathbf{k}\cdot\mathbf{E}=0$. Inserting $\overline{\mathbf{Q}}^{(2)}$ from (18) into (17), one finds that the free-space, time-average Poynting vector in the $\hat{\mathbf{z}}$ direction is given in terms of the time-average Poynting vector for the source-free propagating modes in the continuum by the expression

$$\frac{1}{2}\mathrm{Re}[\mathbf{E}^{(1)}\times\mathbf{H}^{(1)*}]\cdot\hat{\mathbf{z}} = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{\mathbf{z}} - \frac{\omega\alpha_Q\varepsilon_0}{8}k_z|\mathbf{E}|^2 \quad (19)$$

in which the superscripts "(2)" in the continuum have been dropped and higher order terms proportional to $k^2\alpha_Q$ have been neglected because, as mentioned above, $k^2\alpha_Q \ll 1$. Since the free-space Poynting vector $\mathrm{Re}[\mathbf{E}^{(1)}\times\mathbf{H}^{(1)*}]\cdot\hat{\mathbf{z}}/2$ is unequivocally the time-average power per unit area (Pow) entering the continuum in the z direction [4], that is

$$\mathrm{Pow} = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{\mathbf{z}} - \frac{\omega\alpha_Q\varepsilon_0}{8}k_z|\mathbf{E}|^2 \quad (20)$$

we see that the power is not just equal to $\mathrm{Re}\left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{\mathbf{z}}/2$ in the lossless, source-free, spatially dispersive continuum but requires the second term on the right-hand side of (20).

As explained in the Introduction, the power per unit area ($\mathrm{POW}_{\mathrm{arr}}$) in a lossless, reciprocal periodic array has been determined by Silveirinha et al. [2,3] and can be written for the z direction as

$$\mathrm{Pow}_{\mathrm{arr}} = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{\mathbf{z}} - \frac{\omega}{4}\mathbf{E}\cdot\frac{\partial\bar{\boldsymbol{\varepsilon}}(\mathbf{k},\omega)}{\partial k_z}\cdot\mathbf{E}^* \quad (21)$$

where the partial derivative with respect to k_z is taken holding ω fixed. Taking the partial k_z derivative of the lossless reciprocal $\bar{\boldsymbol{\varepsilon}}(\mathbf{k},\omega)$ in (16), we find

$$\frac{\partial\bar{\boldsymbol{\varepsilon}}(\mathbf{k},\omega)}{\partial k_z} = \frac{\alpha_Q\varepsilon_0}{4}(2k_z\bar{\mathbf{I}}+\hat{\mathbf{z}}\mathbf{k}+\mathbf{k}\hat{\mathbf{z}}) \quad (22)$$

which when substituted into (21) gives

$$\mathrm{Pow}_{\mathrm{arr}} = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{\mathbf{z}} - \frac{\omega\alpha_Q\varepsilon_0}{8}k_z|\mathbf{E}|^2 \quad (23)$$

the same power in the lossless spatially dispersive continuum as that determined in (20) from the boundary conditions.

3 Group Velocity of a Wavepacket in a Spatially Dispersive Continuum

The space-time fields in a spatially dispersive continuum can be expressed as a four-fold integral consisting of a 3D Fourier transform of the fields in real $\mathbf{k}$ space [1] and an analytic Fourier transform over positive real angular frequencies ω [10, ch. 5]. Moreover, for any lossless source-free (normal or characteristic) mode (eigenfunction), the positive frequency ω is a unique function of the real propagation vector $\mathbf{k}$ and the frequency Fourier transform collapses to one discrete value, so that the space-time electric field, for example, can be expressed as

$$\mathscr{E}(\mathbf{r},t) = 2\mathrm{Re}\int_{\mathbf{k}^\pm}\mathbf{E}[\mathbf{k},\omega(\mathbf{k})]e^{i[\mathbf{k}\cdot\mathbf{r}-\omega(\mathbf{k})t]}d^3k \quad (24)$$

with $d^3k = dk_xdk_ydk_z$ and the integration limits $\mathbf{k}^\pm$ are $(-\infty,-\infty,-\infty) < (k_x,k_y,k_z) < (+\infty,+\infty,+\infty)$. A wavepacket is defined by assuming the wave propagation vectors are concentrated about a central propagation vector $\mathbf{k}_p$ such that $\mathbf{E}(\mathbf{k},\omega)\approx 0$ unless $|\mathbf{k}-\mathbf{k}_p|/|\mathbf{k}_p| = |\Delta\mathbf{k}|/|\mathbf{k}_p| \ll 1$. Assuming $\omega(\mathbf{k})$ is expandable about $\mathbf{k}_p$, we have

$$\omega(\mathbf{k}) = \omega(\mathbf{k}_p)+\nabla_k\omega(\mathbf{k}_p)\cdot\Delta\mathbf{k}+O\left[|\Delta\mathbf{k}|^2\right] \quad (25)$$

so that $\mathscr{E}(\mathbf{r},t)$ in (24) can be approximated as

$$\mathscr{E}(\mathbf{r},t)\approx 2\mathrm{Re}\Big\{e^{i(\mathbf{k}_p\cdot\mathbf{r}-\omega_pt)}\int_{\Delta\mathbf{k}}\mathbf{E}[\Delta\mathbf{k}+\mathbf{k}_p,\omega(\Delta\mathbf{k}+\mathbf{k}_p)] \cdot e^{i\Delta\mathbf{k}\cdot[\mathbf{r}-\nabla_k\omega(\mathbf{k}_p)t]}d^3\Delta k\Big\} \quad (26a)$$

or simply

$$\mathscr{E}(\mathbf{r},t)\approx \mathrm{Re}\left\{\mathbf{g}[\mathbf{r}-\nabla_k\omega(\mathbf{k}_p)t]e^{i(\mathbf{k}_p\cdot\mathbf{r}-\omega_pt)}\right\} \quad (26b)$$

where $\omega_p = \omega(\mathbf{k}_p)$ and $\mathbf{g}(\mathbf{r})$ is an envelope function that varies slowly over a distance equal to the wavelength $\lambda_p = 2\pi/|\mathbf{k}_p|$ in the medium. The velocity of this envelope or wavepacket is called the group velocity and is determined from (26b) as

$$\mathbf{v}_g(\mathbf{k}_p) = \nabla_k\omega(\mathbf{k}_p) \quad (27)$$

which is not necessarily in the same direction as the phase velocity $\mathbf{v}_p = (\omega_p/|\mathbf{k}_p|)\hat{\mathbf{k}}_p$. For lossless reciprocal material, the group velocities in the $\pm\mathbf{k}_p$ directions are equal [12] and thus only values of $(k_{px},k_{py},k_{pz}) > 0$ as well as $\omega(\mathbf{k}) > 0$ need to be considered for reciprocal material.

4 Energy-Transport Velocity in a Spatially Dispersive Continuum

In this section, a ratio expression is derived for the energy-transport velocity that equals the group velocity of a wavepacket in a source-free, lossless, spatially dispersive, anisotropic medium where $\omega = \omega(\mathbf{k})$. This ratio expression contains the power derived in Section 2.1 as its numerator and, thus, its denominator reveals the stored energy density in a lossless spatially dispersive medium.

Taking the differential of (10a) and (10b) dotted into $\mathbf{H}^*$ and $\mathbf{E}^*$, respectively, one obtains with $\mathbf{J}=0$

$$[d\mathbf{k}\times\mathbf{E}+\mathbf{k}\times d\mathbf{E}-d(\omega\mathbf{B})]\cdot\mathbf{H}^* = 0 \quad (28a)$$

$$[d\mathbf{k}\times\mathbf{H}+\mathbf{k}\times d\mathbf{H}+d(\omega\mathbf{D})]\cdot\mathbf{E}^* = 0. \quad (28b)$$

Subtracting these two equations and then using the constitutive and lossless relations in (11) and (12) for anisotropic media leads to

$$d\mathbf{k}\cdot\mathbf{S}(\mathbf{k},\omega) = U(\mathbf{k},\omega)d\omega = U(\mathbf{k},\omega)d\mathbf{k}\cdot\nabla_k\omega(\mathbf{k}) \quad (29)$$

where

$$\mathbf{S}(\mathbf{k},\omega) = \mathrm{Re}[\mathbf{E}(\mathbf{k},\omega)\times\mathbf{H}^*(\mathbf{k},\omega)]/2 \quad (30)$$

is the time-average Poynting vector and

$$U(\mathbf{k},\omega) = \frac{1}{4}\left[\mathbf{E}\cdot\frac{d[\omega\overline{\boldsymbol{\varepsilon}}(\mathbf{k},\omega)]}{d\omega}\cdot\mathbf{E}^* + \mathbf{H}\cdot\frac{d[\omega\overline{\boldsymbol{\mu}}(\mathbf{k},\omega)]}{d\omega}\cdot\mathbf{H}^*\right] \quad (31)$$

has dimensions of an energy density. However, $U(\mathbf{k},\omega)$ in (31) is not uniquely defined. To see this, evaluate each element of the total derivatives in (31) as

$$\frac{d[\omega\varepsilon_{ij}(\mathbf{k},\omega)]}{d\omega} = \frac{\partial[\omega\varepsilon_{ij}(\mathbf{k},\omega)]}{\partial\omega} + \omega\nabla_k\varepsilon_{ij}(\mathbf{k},\omega)\cdot\frac{d\mathbf{k}}{d\omega} \quad (32)$$

and similarly for μ_{ij}, where $\nabla_k\varepsilon_{ij}(\mathbf{k},\omega)$ is the $\mathbf{k}$ gradient holding ω fixed, and $\partial\varepsilon_{ij}(\mathbf{k},\omega)/\partial\omega$ is the partial ω derivative holding $\mathbf{k}$ fixed (as if $\mathbf{k}$ and ω were independent variables). Although ω is a unique function of $\mathbf{k}$ for a source free mode, the vector $\mathbf{k}$ is not uniquely determined by ω, so that the ratio $d\mathbf{k}/d\omega$ varies with $d\mathbf{k}$ and, thus, $U(\mathbf{k},\omega)$ is not, in general, uniquely defined. Nonetheless with these total derivatives inserted into the first equation in (29), and writing $d\omega(\mathbf{k}) = d\mathbf{k}\cdot\nabla_k\omega(\mathbf{k})$ for source-free modes, one obtains

$$\begin{aligned}4U d\omega = 4d\mathbf{k}\cdot\mathbf{S} = d\mathbf{k}\cdot\nabla_k\omega(\mathbf{k})\Bigg[\mathbf{E}\cdot\frac{\partial[\omega\overline{\boldsymbol{\varepsilon}}(\mathbf{k},\omega)]}{\partial\omega}\cdot\mathbf{E}^* \\ +\mathbf{H}\cdot\frac{\partial[\omega\overline{\boldsymbol{\mu}}(\mathbf{k},\omega)]}{\partial\omega}\cdot\mathbf{H}^*\Bigg] \quad (33) \\ +\omega d\mathbf{k}\cdot\nabla_k\varepsilon_{ij}(\mathbf{k},\omega)E_iE_j^* + \omega d\mathbf{k}\cdot\nabla_k\mu_{ij}(\mathbf{k},\omega)H_iH_j^*.\end{aligned}$$

Since (33) holds for all $d\mathbf{k}$, and all other quantities in (33) are uniquely defined, it follows that

$$\nabla_k\omega(\mathbf{k}) = \mathbf{S}_{\mathrm{tot}}(\mathbf{k},\omega)/W(\mathbf{k},\omega) \quad (34a)$$

with

$$\mathbf{S}_{\mathrm{tot}}(\mathbf{k},\omega) = \mathbf{S}(\mathbf{k},\omega) - \frac{\omega}{4}[\nabla_k\varepsilon_{ij}(\mathbf{k},\omega)E_iE_j^* + \nabla_k\mu_{ij}(\mathbf{k},\omega)H_iH_j^*] \quad (34b)$$

and

$$W(\mathbf{k},\omega) = \frac{1}{4}\left[\mathbf{E}\cdot\frac{\partial[\omega\overline{\boldsymbol{\varepsilon}}(\mathbf{k},\omega)]}{\partial\omega}\cdot\mathbf{E}^* + \mathbf{H}\cdot\frac{\partial[\omega\overline{\boldsymbol{\mu}}(\mathbf{k},\omega)]}{\partial\omega}\cdot\mathbf{H}^*\right]. \quad (34c)$$

The power $\mathbf{S}_{\mathrm{tot}}(\mathbf{k},\omega)$ in the numerator of (34a) is the anisotropic generalization of the total power (energy flow per unit area) in a lossless spatially dispersive continuum found in (19)–(23). Consequently, the energy density $W(\mathbf{k},\omega)$ in the denominator of (34a) is certainly the stored energy density in a lossless, spatially dispersive, anisotropic continuum (as well as in a lossless, spatially nondispersive, anisotropic continuum where $\overline{\boldsymbol{\varepsilon}}$ and $\overline{\boldsymbol{\mu}}$ are not functions of $\mathbf{k}$, and $\mathbf{S} = \mathbf{S}_{\mathrm{tot}}$ is the total power). The ratio $\mathbf{S}_{\mathrm{tot}}/W$ is in the form of an "energy-transport velocity" $\mathbf{v}_e(\mathbf{k}_p)$ equal to the group velocity. These equalities can be summarized in the one extended equation

$$\mathbf{v}_g(\mathbf{k}_p) = \nabla_k\omega(\mathbf{k}_p) = \mathbf{v}_e(\mathbf{k}_p) = \mathbf{S}_{\mathrm{tot}}(\mathbf{k}_p,\omega_p)/W(\mathbf{k}_p,\omega_p). \quad (35)$$

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